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Kraichnan Flow in a Square: An Example of Integrable Chaos

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The Kraichnan flow provides an example of a random dynamical system accessible to an exact analysis. We study the evolution of the infinitesimal separation between two Lagrangian trajectories of the flow. Its long-time asymptotics is reflected in the large deviation regime of the statistics of stretching exponents. Whereas in the flow that is isotropic at small scales the distribution of such multiplicative large deviations is Gaussian, this does not have to be the case in the presence of an anisotropy. We analyze in detail the flow in a two-dimensional periodic square where the anisotropy generally persists at small scales. The calculation of the large deviation rate function of the stretching exponents reduces in this case to the study of the ground state energy of an integrable periodic Schrödinger operator of the Lamé type. The underlying integrability permits to explicitly exhibit the non-Gaussianity of the multiplicative large deviations and to analyze the time-scales at which the large deviation regime sets in. In particular, we indicate how the divergence of some of those time scales when the two Lyapunov exponents become close allows a discontinuity of the large deviation rate function in the parameters of the flow. The analysis of the two-dimensional anisotropic flow permits to identify the general scenario for the appearance of multiplicative large deviations together with the restrictions on its applicability.

KEY WORDS: multiplicative large deviations, Kraichnan model

1. INTRODUCTION

The Kraichnan random ensemble of velocities$^{(23)}$ has been extensively used to model various phenomena related to turbulent transport both in the inertial interval of scales that develops at high Reynolds numbers and at moderate Reynolds numbers where the viscosity effects play an important role.$^{(16)}$ The passive transport

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of scalar or vector quantities in a velocity field is governed by the Lagrangian flow describing the evolution of the trajectories of fluid particles. From the mathematical point of view, such flow provides an example of a random dynamical system[1] which in the case of the Kraichnan velocities is described by a stochastic differential equation. For the Kraichnan flows corresponding to moderate Reynolds numbers, the methods borrowed from the theory of random dynamical systems or stochastic differential equations appear to provide important information about the transport properties of the flows. To start with, the values of the Lyapunov exponents of the flow, whose existence is asserted by the multiplicative ergodic theorem, allow to decide whether the flow is chaotic (positive top Lyapunov exponent) or not, leading to different directions of the cascades of passively advected scalars.[12] More detailed information about the transport properties of the flow may be extracted from the knowledge of the fluctuations of the exponential stretching rates around their limiting long-time values equal to the Lyapunov exponents. In the generic case where all the Lyapunov exponents are different, the statistics of the stretching exponents may be expected to exhibit at long but finite times a large deviation regime captured quantitatively by a single function of the vector of the stretching rates. Since the existence of such multiplicative large deviation regime is not assured by general mathematical theorems, see Ref. 5 for partial results, it is interesting to have at our disposal models where it may be established and studied in detail.

One such example that has been known for some time is the homogeneous isotropic Kraichnan flow. The corresponding stochastic differential equation has been studied in the mathematical literature in the eighties and nineties of the last century. In particular, the Lyapunov exponents have been found in Ref. 25 and 4 but the large deviations have been studied only separately for the top stretching exponent or for the sum of them.[5] With the regain of interest of physicists in the Kraichnan model in the mid-nineties, the same stochastic equation resurfaced as the model for the Lagrangian flow at moderate Reynolds numbers with the motivations, the accents and the language proper to the turbulence theory community.[10] In particular, it was realized that many properties of the turbulent transport require more information about the flow than the spectrum of Lyapunov exponents and may be expressed in terms of the rate function of the large deviations of the stretching exponents. Those include the rate of decay of the moments of advected scalar[3] or of growth of those of the magnetic field[11] or, in compressible flows, of the density fluctuations,[2] the multifractal properties of long-time density concentrations[6] and the threshold for the onset of the drag reduction in polymer solutions.[9] In the homogeneous and isotropic Kraichan flow, the large deviation regime of the stretching exponents is Gaussian and the corresponding rate function is a quadratic polynomial.[3,2] Its simple form has permitted to extract analytic answers for many characteristics features of passive advection in such flows.[16]

The simplicity of the multiplicative large deviation regime in the homogeneous isotropic Kraichnan flow is due to the decoupling of the dynamics of the stretching exponents from that of the eigen-directions for stretching and contraction.[(3)] As a result also the exact distribution of the stretching exponents may be found analytically in this case as it appears to be related to the heat kernel of the integrable quantum Calogero-Sutherland Hamiltonian for particles on the line interacting with the attractive pair potential proportional to the function $\sinh^{-2}$ of the inter-particle distance.[(7,19)] In the present paper we analyze the two-dimensional Kraichnan flow in a periodic square often used in numerical simulations. The large scale anisotropy due to the shape of the flow volume generically persists on small scales inducing isotropy breaking terms in the distribution of strain that drives the evolution of the stretching exponents. Due to the presence of such terms, the stretching exponents dynamics does not decouple anymore from that of the (unstable) eigen-directions. The Lyapunov exponents may nevertheless be still computed analytically and their difference expressed in terms of elliptic integrals. The distribution of the sum of the stretching exponents is still Gaussian for all times (this is a general fact for the homogeneous Kraichnan flows). As for the rate function of the large deviations of the difference of the stretching exponents, its calculation may be reduced to that of the ground-state energy of the integrable one-dimensional periodic quantum Lamé operator. For general values of the coupling constant, the eigenvalues of the latter may be found by numerical diagonalization of infinite tridiagonal matrices.[(14)] Those matrices reduce to finite ones at integer values of the coupling and for the lowest eigenvalues. Alternatively, the ground state energy of the Lamé operator may be found by direct numerical integration of the eigenvalue equation. Both approaches permit to obtain the large deviation rate function for the stretching exponents that turns out to be non-quadratic although with quadratic asymptotes. The analysis of the spectral gap of the Lamé operator permits to assess the time scales at which the multiplicative large deviation regime sets in. In particular, the divergence of the time scales relative to the multiplicative central-limit regime when the difference of the Lyapunov exponents tends to zero accompanies the observed discontinuity of the large deviation rate function in the anisotropy parameter at the point where the Lyapunov exponents coincide.

The plan of the paper is as follows. In Sect. II, we discuss the relations between the Lagrangian flow and random dynamical systems introducing the concepts of the natural invariant measure and of the tangent process and stating two different definitions of the stretching exponents. In Sect. III we introduce the Kraichnan ensemble of velocities and discuss how the general concepts considered before simplify in the homogeneous Kraichnan velocities. We recall briefly the results about the statistics of the stretching exponents in the isotropic version of the Kraichnan model. Section IV is the core of the present paper. We discuss there the Kraichnan flow in a periodic square, the persistence of anisotropy at small distances, the calculation of the Lyapunov exponents and, finally, the large deviations for the stretching exponents, their relation to the Lamé equation, their non-Gaussianity, and their discontinuity in the anisotropy parameters. The last section collects our conclusions. We believe that the simple model considered here allows to identify a general scenario for the occurrence of the multiplicative large deviations when all the Lyapunov exponents are different and to understand the mechanism of its failure when some of the Lyapunov exponents get close.

2. LAGRANGIAN FLOW AS A RANDOM DYNAMICAL SYSTEM

2.1. Random Dynamical Systems

We shall start by describing the Lagrangian flow in the language of random dynamical systems. Let us consider an ensemble of velocities $\boldsymbol{u}_t^{\omega}(\boldsymbol{r})$ in a bounded region V of the d-dimensional space. Here ω is a random parameter belonging to a probability space Ω equipped with a probability measure $P(d\omega)$. Ω may be taken as the space of velocity realizations. We shall assume the stationarity of the velocity ensemble, i.e. the existence of a 1-parameter group of measure preserving transformations $\omega \mapsto \omega_s$ of Ω such that $\boldsymbol{u}_t^{\omega_s}(\boldsymbol{r}) = \boldsymbol{u}_{t+s}^{\omega}(\boldsymbol{r})$. The Lagrangian flow describing the trajectories of tracer particles carried by the fluid is defined by the ordinary differential equation

$$\frac{d\boldsymbol{R}}{dt} = \boldsymbol{u}_t^{\omega}(\boldsymbol{R}). \tag{1}$$

Under simple regularity assumptions including the spatial smoothness of the velocities $\boldsymbol{u}_t^{\omega}(\boldsymbol{r})$, the solutions $\mathbf{R}_t^{\omega}(\boldsymbol{r})$ of Eq. (1) parametrized by their time zero position $\boldsymbol{r}$ define a family of random smooth maps Φ_t^{ω} of the region V such that $\mathbf{R}_t^{\omega}(\boldsymbol{r}) = \Phi_t^{\omega}(\boldsymbol{r})$ with the composition rule

$$\boldsymbol{\Phi}_{s+t}^{\omega} = \boldsymbol{\Phi}_t^{\omega_s} \circ \boldsymbol{\Phi}_s^{\omega}. \tag{2}$$

In particular, one obtains a 1-parameter group of transformations

$$(\boldsymbol{r},\, \omega) \longmapsto \big(\Phi_t^{\omega}(\boldsymbol{r}),\, \omega_t\big) \tag{3}$$

of the product space $V \times \Omega$ which realizes the flow dynamics.

2.2. Natural Invariant Measure

Note that $\Phi_{-s}^{\omega_s}(\boldsymbol{r})$ is the time zero position of the solution that at time s passes through $\boldsymbol{r}$. Suppose that, for continuous functions f on V, the limit

$$\lim_{s\to-\infty} \frac{1}{|V|} \int_V f\big(\Phi_{-s}^{\omega_s}(\boldsymbol{r})\big)\, d\boldsymbol{r} =: \int_V f(\boldsymbol{r})\, \mu^{\omega}(d\boldsymbol{r}) \tag{4}$$

exists for almost all ω and defines a random family of probability measures $\mu^\omega(d\boldsymbol{r})$ on V. Note that due to the composition rule (2),

$$\int_V f\big(\Phi_t^\omega(\boldsymbol{r})\big)\,\mu^\omega(d\boldsymbol{r}) \;=\; \int_V f(\boldsymbol{r})\,\mu^{\omega_t}(d\boldsymbol{r})\,. \tag{5}$$

The measures $\mu^\omega(d\boldsymbol{r})$ describe the distribution of the time zero positions of the Lagrangian trajectories whose initial points were uniformly seeded in the far past. For incompressible velocities the uniform distribution is conserved by the flow so that $\mu^\omega(d\boldsymbol{r}) = \frac{1}{|V|}d\boldsymbol{r}$ but in the presence of compressibility, the Lagrangian trajectories develop preferential concentrations and the measures $\mu^\omega(d\boldsymbol{r})$ tend to be singular and supported by lower dimensional random attractors.

From the random measures $\mu^\omega(d\boldsymbol{r})$ one may synthesize a measure $M(d\boldsymbol{r}, d\omega)$ on the product space $V \times \Omega$ defined by the relation

$$\int_{V\times\Omega} f(\boldsymbol{r},\omega)\,M(d\boldsymbol{r},d\omega) \;=\; \left\langle \int_V f(\boldsymbol{r},\omega)\,\mu^\omega(d\boldsymbol{r})\right\rangle$$

where $\langle\cdot\cdot\rangle$ denotes the expectation w.r.t. the probability measure $P(d\omega)$. Note that due to the property (5),

$$\int_{V\times\Omega} f\big(\Phi_t^\omega(\boldsymbol{r}),\omega_t\big)\,M(d\boldsymbol{r},d\omega) \;=\; \int_{V\times\Omega} f(\boldsymbol{r},\omega)\,M(d\boldsymbol{r},d\omega)$$

so that the measure $M(d\boldsymbol{r}, d\omega)$ is invariant under the 1-parameter group (3) of dynamical transformations. We shall call $M(d\boldsymbol{r}, d\omega)$ the **natural invariant measure** of the random dynamical system (1). Below, we shall assume that $M(d\boldsymbol{r}, d\omega)$ is ergodic with respect to the group (3), i.e. that functions invariant under the dynamics are constant M-almost everywhere.

2.3. Tangent Process

Much information about the Lagrangian flow may be extracted by looking at the evolution of the separation between two infinitesimally close trajectories. Consider the Jacobi matrix $\nabla\Phi_t^\omega(\boldsymbol{r}) \equiv W_t^\omega(\boldsymbol{r})$ with the entries

$$W^i_{\ j} \;=\; \nabla_j \Phi_t^{\omega i}(\boldsymbol{r})\,.$$

The matrix $W_t^\omega(\boldsymbol{r})$ propagates the infinitesimal separations:

$$\delta\boldsymbol{R}^\omega(t;\boldsymbol{r}) \;=\; W_t^\omega(\boldsymbol{r})\,\delta\boldsymbol{r}\,.$$

Note that $W_0^\omega(\boldsymbol{r}) = Id$ and that $W_t^\omega(\boldsymbol{r})$ satisfies the linear differential equation

$$\frac{dW}{dt} \;=\; S_t^\omega\, W \tag{6}$$

with $(S_t^\omega)^i{}_j = \nabla_j u_t^{\omega i}(\boldsymbol{R}_t^\omega(\boldsymbol{r}))$ equal to the matrix elements of the strain along the Lagrangian trajectory. We shall call $W_t^\omega(\boldsymbol{r})$ the **tangent process**. In studying it below, Eq. (6) will play a crucial role. Note that the composition rule (2) implies that

$$W_{s+t}^\omega(\boldsymbol{r}) \;=\; W_t^{\omega_s}\big(\Phi_s^\omega(\boldsymbol{r})\big)\, W_s^\omega(\boldsymbol{r})\,.$$

In particular,

$$W_{-t}^\omega(\boldsymbol{r}) \;=\; W_t^{\omega_{-t}}\big(\Phi_{-t}^\omega(\boldsymbol{r})\big)^{-1}\,. \tag{7}$$

We shall be interested in the statistical properties of the $d \times d$ matrices $W_t^\omega(\boldsymbol{r})$ for fixed but large t with $(\boldsymbol{r}, \omega)$ sampled according to the natural invariant measure $M(d\boldsymbol{r}, d\omega)$. As for any real invertible $d \times d$ matrix, one may decompose

$$W \;=\; O' \,\mathrm{diag}[\mathrm{e}^{\rho_1}, \ldots, \mathrm{e}^{\rho_d}]\; O \tag{8}$$

with a diagonal positive definite matrix sandwiched in between orthogonal ones. One may demand that the **stretching exponents** $\rho_i = \rho_{i\,t}^{\omega}(\boldsymbol{r})$ given by half the logarithm of the eigenvalues of the matrices $W^T W$ and $W W^T$, be ordered so that $\rho_1 \geq \cdots \geq \rho_d$. They carry an important part of the information about the tangent process. The joint probability distribution function (PDF) of the time t stretching exponents is given by the formula:

$$P_t(\boldsymbol{\rho}) \;=\; \int_{V\times\Omega} \prod_{i=1}^{d} \delta\big(\rho_i - \rho_{i\,t}^{\omega}(\boldsymbol{r})\big)\, M(d\boldsymbol{r}, d\omega)\,.$$

Note the relation

$$\rho_{i\,-t}^{\omega}(\boldsymbol{r}) \;=\; -\rho_{d-i+1\,t}^{\omega_{-t}}\big(\Phi_{-t}^\omega(\boldsymbol{r})\big)$$

between the forward and the backward exponents that follows from Eq. (7). The invariance of the natural measure under the 1-parameter dynamics (3) implies then that

$$P_{-t}(\rho_1, \ldots, \rho_d) \;=\; P_t(-\rho_d, \ldots, -\rho_1)\,.$$

2.4. Multiplicative Ergodic Theorem and Multiplicative Large Deviations

The main general result about dynamical systems, the multiplicative ergodic theorem of Oseledec,[26] see also Ref. 27, states that under mild assumptions, the limits

$$\lim_{t\to\pm\infty} \frac{1}{t}\, \ln(W^T W)_t^\omega(\boldsymbol{r}) \;=:\; \Lambda_\pm^\omega(\boldsymbol{r})$$

exist for M-almost all $(\boldsymbol{r}, \omega)$. Besides,

$$\Lambda^{\omega}_{\pm}(\boldsymbol{r}) \;=\; O^{\omega}_{\pm}(\boldsymbol{r})\,\mathrm{diag}[\lambda_1, \cdots, \lambda_d]\, O^{\omega}_{\pm}(\boldsymbol{r})^T \tag{9}$$

where $\lambda_1 \geq \cdots \geq \lambda_d$ are the Lyapunov exponents that, due to the ergodicity assumption, are $(\boldsymbol{r}, \omega)$-independent. In particular, λ_i are the limits when $t \to \infty$ of the ratios $\rho^{\omega}_{i\,t}(\boldsymbol{r})/t$ for M-almost all $(\boldsymbol{r}, \omega)$.

When all Lyapunov exponents are different, one may expect that for large but finite time the distribution of the stretching exponents takes for $\rho_1 > \cdots > \rho_d$ the large deviation form:[3,19]

$$P_t(\boldsymbol{\rho}) \;\propto\; \mathrm{e}^{-t\,H(\rho_1/t,\ldots,\rho_d/t)} \tag{10}$$

with a convex rate function H attaining its minimal value equal to zero at the vector $\boldsymbol{\lambda}$ of the Lyapunov exponents. In particular, if H is regular around $\boldsymbol{\lambda}$ one would obtain, as a corollary, the multiplicative central limit result stating that $\frac{1}{\sqrt{t}}(\boldsymbol{\rho}_t - \boldsymbol{\lambda}t)$ tends when $t \to \infty$ to the vector of normal variables with the inverse covariance given by the second derivative matrix $H''(\boldsymbol{\lambda})$. To our knowledge, no general theorems assure existence of the multiplicative large deviation regime, see Ref. 5 for some partial results. There are, nevertheless, examples of random (and non-random[22]) dynamical systems where the relation (10) indeed holds. As has been already mentioned in Introduction, the examination of one of such examples is the main topic of the present paper.

For time reversible velocity ensembles, i.e. when the velocities $\boldsymbol{u}^{\omega}_t(\boldsymbol{r})$ and $-\boldsymbol{u}^{\omega}_{-t}(\boldsymbol{r})$ have the same distribution, the rate function H possesses the symmetry:

$$H(-\rho_d/t, \ldots, -\rho_1/t) \;=\; H(\rho_1/t, \ldots, \rho_d/t) \;-\; \sum_{i=1}^{d} \rho_i/t\,, \tag{11}$$

see Refs. 2,16. Eq. (11) is an extension, somewhat in the spirit of Ref. 17, of the fluctuation relations considered originally by Evans-Cohen-Morriss[15] and Gallavotti-Cohen[18] for the large-deviation rate function of the phase-space contraction rate $-\sum \rho_i$ and established in a setup of random dynamical systems close to the one of the present paper in Ref. 8.

2.5. Stretching Along the Unstable Flags

We shall call a family $F = (E_i)_{i=1}^{d}$ of i-dimensional subspaces

$$\{0\} \;\subset\; E_1 \;\subset\; \cdots \;\subset\; E_{d-1} \;\subset\; E_d = \mathbf{R}^d$$

a **flag**. An example is provided by the flag F^0 composed by the subspaces E^0_i spanned by the first i vectors of the canonical basis of $\mathbf{R}^d$. Clearly, the group

$GL(d)$ acts on the space of flags. The subgroup $B \subset GL(d)$ composed of the upper triangular matrices preserves the flag F_0 and the space Fl of all flags, the flag variety, may be identified with the homogeneous space $GL(d)/B = O(d)/D$ where $D = O(d) \cap B$ is the subgroup of the diagonal matrices with entries ± 1. We shall denote by dF the normalized $O(d)$-invariant measure on Fl.

The flow Φ_t^ω on the volume V induces the flow Ψ_t^ω on the product space $V \times Fl$ defined by

$$\Psi_t^\omega(\boldsymbol{r}, F) \;=\; \big(\Phi_t^\omega(\boldsymbol{r}),\, W_t^\omega(\boldsymbol{r})F\big)\,.$$

Mimicking the constructions from Sect. 2.2 of the natural invariant measure, one may define the measures $d\sigma^\omega(d\boldsymbol{r}, dF)$ on $V \times Fl$ by the relation

$$\lim_{s\to-\infty}\,\frac{1}{|V|}\int_{V\times Fl} f\big(\Psi_{-s}^{\omega_s}(\boldsymbol{r}, F)\big)\,d\boldsymbol{r}\,dF \;=:\; \int_{V\times Fl} f(\boldsymbol{r}, F)\,\sigma^\omega(d\boldsymbol{r}, dF)$$

if the limit exists for continuous functions f for almost all ω. Clearly, the measures σ^ω depend only on the past velocities and the formula

$$\int_{V\times Fl} f\big(\Psi_t^\omega(\boldsymbol{r}, F)\big)\,\sigma^\omega(d\boldsymbol{r}, dF) \;=\; \int_{V\times Fl} f(\boldsymbol{r}, F)\,\sigma^{\omega_t}(d\boldsymbol{r}, dF)\,. \tag{12}$$

analogous to Eq. (5) holds. Out of the measures σ^ω, one may synthesize a measure $\Sigma(d\boldsymbol{r}, dF, d\omega)$ on the product space $V \times Fl \times \Omega$ by the relation

$$\int_{V\times Fl\times\Omega} f(\boldsymbol{r}, F, \omega)\,\Sigma(d\boldsymbol{r}, dF, d\omega) \;=\; \left\langle\int_{V\times Fl} f(\boldsymbol{r}, F, \omega)\,\sigma^\omega(d\boldsymbol{r}, dF)\right\rangle.$$

For $F = OF^0$ with $O \in O(d)$, consider the Iwasawa decomposition of the invertible matrix $\tilde W = W_t^\omega(\boldsymbol{r})\,O$:

$$\tilde W \;=\; O'\,\mathrm{diag}[\mathrm{e}^{\eta_1}, \ldots, \mathrm{e}^{\eta_d}]\,N$$

with O' orthogonal and N upper-triangular with units on the diagonal. The exponents $\eta_i = \eta^\omega_{i\,t}(\boldsymbol{r}, F)$ do not depend on the freedom in the choice of O. We shall call them the **stretching exponents along the flag** F. With $(\boldsymbol{r}, F, \omega)$ sampled w.r.t. the probability measure $\Sigma(d\boldsymbol{r}, dF, d\omega)$, they become random variables. Their joint time t PDF will be denoted $Q_t(\boldsymbol{\eta})$.

When all the Lyapunov exponents are different then the orthogonal matrices $O_\pm^\omega(\boldsymbol{r})$ in Eq. (9) are determined modulo the right multiplication by matrices from D. In particular, $O_-^\omega(\boldsymbol{r})$ defines a flag $F^\omega(\boldsymbol{r})$ of subspaces $E_i^\omega(\boldsymbol{r}) = O_-^\omega(\boldsymbol{r})E_i^0$ of (less and less) unstable directions of the flow. In this case,

$$\int_{V\times Fl} f(\boldsymbol{r}, F)\,\sigma^\omega(d\boldsymbol{r}, dF) \;=\; \int_V f(\boldsymbol{r}, F^\omega(\boldsymbol{r}))\,\mu^\omega(d\boldsymbol{r})\,,$$

i.e. the measure $\sigma^\omega(d\boldsymbol{r}, dF)$ is concentrated in the direction of the flag variety Fl on the unstable flags. The subspaces $E_i^\omega(\boldsymbol{r})$ may be characterized by the property that for $0 \neq e \in E_i^\omega(\boldsymbol{r}) \setminus E_{i-1}^\omega(\boldsymbol{r})$,

$$\lim_{t \to -\infty} \frac{1}{t} \ln \| W_t^\omega(\boldsymbol{r}) e \| = \lambda_i$$

with the limit testing the far past asymptotics. The unstable flags $F^\omega(\boldsymbol{r})$ depend only on the velocities at the negative times and are covariant under the 1-parameter dynamics (3):

$$W_t^\omega(\boldsymbol{r}) \, F^\omega(\boldsymbol{r}) = F^{\omega_t}\big(\Phi_t^\omega(\boldsymbol{r})\big).$$

We shall call the exponents $\eta_{i\,t}^\omega(\boldsymbol{r}) = \eta_{i\,t}^\omega(\boldsymbol{r}, F^\omega(\boldsymbol{r}))$ the **stretching exponents along the unstable flags**.

The exponents $\eta_{i\,t}^\omega(\boldsymbol{r})$ are not equal to the stretching exponents $\rho_{i\,t}^\omega(\boldsymbol{r})$ introduced previously. In particular, they are not necessarily non-increasing with i although again the ratios $\eta_{i\,t}^\omega(\boldsymbol{r})/t$ tend for M-almost all $(\boldsymbol{r}, \omega)$ to the ordered Lyapunov exponents λ_i when $t \to \pm\infty$. Although the PDFs $Q_t(\boldsymbol{\eta})$ and $P_t(\boldsymbol{\rho})$ are, in general, different (in particular, the latter is non-zero only for ordered arguments), we shall see below that the large deviation parts of P_t and Q_t are closely related. In fact, the stretching exponents η_i are more natural objects than the exponents ρ_i and, as we shall see in examples, their evolution is often simpler to describe.

3. LAGRANGIAN FLOW IN THE KRAICHNAN MODEL

3.1. Kraichnan Ensemble of Velocities

The Kraichnan ensemble of d-dimensional velocities $\boldsymbol{u}^\omega(t, \boldsymbol{r})$ is the Gaussian random ensemble characterized by vanishing mean and time decorrelated covariance:

$$\left\langle u_t^{\omega i}(\boldsymbol{r}) \, u_{t'}^{\omega j}(\boldsymbol{r}') \right\rangle = \delta(t - t') \, D^{ij}(\boldsymbol{r}, \boldsymbol{r}').$$

In particular, we shall consider a homogeneous ensemble in the d-dimensional periodic box V of side L with the spatial covariance given by the Fourier series:

$$D^{ij}(\boldsymbol{r}, \boldsymbol{r}') = \sum_{\boldsymbol{k} \in \frac{2\pi}{L} \mathbf{Z}^d} [(1 - \wp)\delta^{ij} - (1 - \wp d) k^i k^j] \, e^{i \boldsymbol{k} \cdot (\boldsymbol{r} - \boldsymbol{r}')} \, \hat{d}(|\boldsymbol{k}|), \tag{13}$$

as often used in numerical simulations. The spectral function $\hat{d}$ will be assumed fast decreasing or of compact support so that the resulting covariance and, consequently, almost all velocity realizations $\boldsymbol{u}_t^\omega(\boldsymbol{r})$ are smooth in space. The simplest would be to take $\hat{d}$ supported only on the modes with $|\boldsymbol{k}| = \frac{2\pi}{L}$, i.e. on the lowest nontrivial ones. The parameter $\wp$ in Eq. (13) is called the **compressibility degree**. It is equal to the ratio of the covariances $\langle(\Sigma\nabla_i u^{\omega i})^2\rangle/\langle\Sigma(\nabla_i u^{\omega j})^2\rangle$ and is contained between zero and one. Vanishing $\wp$ corresponds to an incompressible flow and $\wp = 1$ to a gradient one.

The Lagrangian flow in the Kraichnan ensemble of velocities is defined by the ordinary differential Eq. (1). Because of the white-noise temporal behavior of the Kraichnan velocities, Eq. (1) becomes, however, a stochastic differential equation and, in line with more standard notations, will be written in the form

$$d\boldsymbol{R} \;=\; \boldsymbol{u}_t^{\omega}(\boldsymbol{r})\,dt\,.$$

In principle, it requires a choice of the stochastic convention, like Itô's or Stratonovich's one, but in the case in question both choices lead to the same solutions (this is due to the vanishing of $\nabla_{r^j} D^{ij}(\boldsymbol{0},\boldsymbol{0})$). As before, one obtains from the solutions a family Φ_ω^t of smooth random maps.[24]

For the Kraichnan model, the convergence (4) takes place in the L^2 norm of the Gaussian process. Besides, due to the homogeneity of the velocity ensemble,

$$\int_{V\times\Omega} f(\boldsymbol{r})\,M(d\boldsymbol{r},d\omega) \;=\; \left\langle \int_V f(\boldsymbol{r})\,\mu^{\omega}(d\boldsymbol{r})\right\rangle \;=\; \frac{1}{|V|}\int_V f(\boldsymbol{r})\,d\boldsymbol{r}\,. \qquad (14)$$

Similarly, due to the homogeneity,

$$\begin{aligned}\int_{V\times Fl\times\Omega} f(\boldsymbol{r},F)\;\Sigma(d\boldsymbol{r},dF,d\omega) &= \left\langle \int_{V\times Fl} f(\boldsymbol{r},F)\,\sigma^{\omega}(d\boldsymbol{r},dF)\right\rangle \\ &= \frac{1}{|V|}\int_{V\times Fl} f(\boldsymbol{r},F)\,d\boldsymbol{r}\,\chi(dF) \qquad (15)\end{aligned}$$

for some probability measure $\chi(dF)$ on Fl. Note that averaging Eq. (12) with respect to the probability measure $P(d\omega)$ for functions f independent of $\boldsymbol{r}$, one infers that

$$\begin{aligned}\int_{Fl}\big\langle f\big(W_t^{\omega}(\boldsymbol{r}_0)F\big)\big\rangle\,\chi(dF) &= \frac{1}{|V|}\int_{V\times Fl}\big\langle f\big(W_t^{\omega}(\boldsymbol{r})F\big)\big\rangle\,d\boldsymbol{r}\,\chi(dF) \\ &= \int_{Fl} f(F)\,\chi(dF)\,, \qquad (16)\end{aligned}$$

i.e. that the measure $\chi(dF)$ is invariant under the process W_t^{ω}.

3.2. Tangent Process in Kraichnan Velocities

Further simplifications appear in the statistics of the tangent process $W_t^{\omega}(\boldsymbol{r})$. For positive t, $W_t^{\omega}(\boldsymbol{r})$ depends only on the velocities at positive times and $\mu^{\omega}(\boldsymbol{r})$ on the velocities at negative times. The temporal decorrelation of the Kraichnan velocities implies then for any function f of invertible $d \times d$ matrices and for $t \geq 0$ the factorization

$$\int_{V\times\Omega} f\big(W_t^\omega(\boldsymbol{r})\big)\ M(d\boldsymbol{r}, d\omega) \ \equiv\ \Big\langle \int_V f\big(W_t^\omega(\boldsymbol{r})\big)\ \mu^\omega(d\boldsymbol{r})\Big\rangle$$
$$= \Big\langle \int_V \big\langle f\big(W_t^\omega(\boldsymbol{r})\big)\big\rangle\, \mu^\omega(d\boldsymbol{r})\Big\rangle \ =\ \frac{1}{|V|}\int_V \big\langle f\big(W_t^\omega(\boldsymbol{r})\big)\big\rangle\, d\boldsymbol{r} \ =\ \big\langle f\big(W_t^\omega(\boldsymbol{r}_0)\big)\big\rangle$$

where the last but one equality follows from the relation (14) and the last one is again due to the homogeneity of the velocity ensemble. We infer that it is enough to know the distribution of $W_t^\omega(\boldsymbol{r}_0)$ for one fixed $\boldsymbol{r}_0$. This simplifies considerably the analysis of the statistics of the stretching exponents in the Kraichnan model.

Let us suppose now that f is a function on $GL(d)$ invariant under the right multiplication of its argument by diagonal matrices with entries ± 1 so that $f(\tilde{W})$ for $\tilde{W} = W_t^\omega(\boldsymbol{r})\, O$ depends on O only via the flag $F = OF^0$ (but is not, in general, a function of $W_t^\omega(\boldsymbol{r})F$ only), see Sect. 2.5. Again due to the decorrelation of velocities at positive and negative times and the relation (15),

$$\int_{V\times Fl\times\Omega} f(\tilde{W})\ \Sigma(d\boldsymbol{r}, dF, d\omega) \ \equiv\ \Big\langle \int_{V\times Fl} f\big(W_t^\omega(\boldsymbol{r})O\big)\ \sigma_\omega(d\boldsymbol{r}, dF)\Big\rangle$$
$$= \frac{1}{|V|}\int_{V\times Fl} \big\langle f\big(W_t^\omega(\boldsymbol{r})\, O\big)\big\rangle\, d\boldsymbol{r}\ \chi(dF) \ =\ \int_{Fl} \big\langle f\big(W_t^\omega(\boldsymbol{r}_0)\, O\big)\big\rangle\ \chi(dF) \tag{17}$$

The last relation permits to simplify the analysis of the statistics of the stretching exponents along the unstable flags.

3.3. Multiplicative Stochastic Equation for the Tangent Process

For fixed $\boldsymbol{r}_0$, the distribution of the tangent process $W_t^\omega(\boldsymbol{r}_0)$ may be obtained by solving the multiplicative stochastic equation

$$dW \ =\ S_t^\omega\, W dt\,, \tag{18}$$

the stochastic version of Eq. (6), with initial condition $W_0 = Id$. The further crucial simplification, due to the time decorrelation and spatial homogeneity of the Kraichnan velocity ensemble, is that in Eq. (18) one may take $(S_t^\omega)^i{}_j = \nabla_j u_t^{\omega i}(\boldsymbol{r}_0)$ instead of $(S_t^\omega)^i{}_j = \nabla_j u_t^{\omega i}(\boldsymbol{R}_t^\omega(r_0))$. In other words, the dependence on the trajectory $R_t^\omega(\boldsymbol{r}_0)$ may be dropped from S_t^ω as long as we are interested in the distribution of $W_t^\omega(\boldsymbol{r}_0)$ for fixed $\boldsymbol{r}_0$, provided we consider the differential Eq. (6) with the Itô convention (here the convention does matter, see Appendix A in Ref. 16).

Summarizing, the strain process S^ω_t in Eq. (18) may be taken as the matrix-valued white noise with mean zero and the covariance

$$\left\langle \left(S^\omega_t\right)^i_{\ k} \left(S^\omega_{t'}\right)^j_{\ l}\right\rangle \;=\; \delta(t-t')\,\nabla_{r^k}\nabla_{r'^l} D^{ij}(\mathbf{0},\mathbf{0}) \;=:\; \delta(t-t')\,C^{ij}_{kl}\,. \tag{19}$$

For the spatial velocity covariance $D^{ij}(\boldsymbol{r},\boldsymbol{r}')$ given by the Fourier series (13),

$$\begin{aligned} C^{ij}_{kl} &= \sum_{\vec{k}\in\frac{2\pi}{L}\mathbf{Z}^d} [(1-\wp)\delta^{ij} - (1-\wp d)k^i k^j]k_k k_l\,\hat{d}(|\vec{k}|) \\ &= 2\alpha\,\delta^{ij}_{kl} \;+\; \beta\,(\delta^i_k\delta^j_l + \delta^i_l\delta^j_k) \;+\; \gamma\,\delta^{ij}\delta_{kl} \end{aligned} \tag{20}$$

where $k^i \equiv k_i$ and δ^{ij}_{kl} is equal to 1 if $i=j=k=l$ and to zero otherwise. The compressibility degree

$$\wp \;=\; \frac{\langle(\mathrm{tr}\,S^\omega)^2\rangle}{\langle\mathrm{tr}\,S^{\omega T} S^\omega\rangle} \;=\; \frac{2\alpha+(d+1)\beta+\gamma}{2\alpha+2\beta+d\gamma}\,.$$

The positivity of the covariance imposes the inequalities

$$\gamma \geq |\beta|\,, \quad 4\alpha+(d+2)\beta+2\gamma \geq d|\beta|\,.$$

We shall exclude the trivial case $\alpha=\beta=\gamma=0$. The case of vanishing α corresponds to the isotropic situation when the distributions of S^ω and of $OS^\omega O^T$ coincide for any orthogonal matrix O. The α-term breaks the $O(d)$-invariance of the distribution of S^ω to the discrete subgroup of the symmetries of a cube. It is the source of a small scale anisotropy that occurs generically in the Kraichnan flow in a periodic box. Indeed, vanishing of α requires a fine tuning of the spectral density $\hat{d}(|\boldsymbol{k}|)$. For example, when $\hat{d}$ is non-zero only for $|\boldsymbol{k}| = \frac{2\pi}{L}$ then necessarily $\alpha \neq 0$.

3.4. Generator of the Tangent Process

The generator of the process W_t satisfying stochastic equation Eq. (18), i.e. the operator $\mathcal{L}$ such that for any regular function f on the group $GL(d)$,

$$\frac{d}{dt}\langle f(W)\rangle \;=\; \langle(\mathcal{L}f)(W)\rangle\,,$$

is given by the formula

$$\begin{aligned} \mathcal{L} &= \tfrac{1}{2}\sum_{\substack{i,j,k,l,\\ n,m=1}}^{d} \Big(2\alpha\,\delta^{ij}_{kl} \;+\; \beta\left(\delta^i_{\ k}\delta^j_{\ l} \;+\; \delta^i_{\ l}\delta^j_{\ k}\right) \;+\; \gamma\,\delta^{ij}\delta_{kl}\Big)\, W^k_{\ m} W^l_{\ n}\,\partial_{W^i_{\ m}}\partial_{W^j_{\ n}} \\ &= \alpha\sum_{i=1}^{d}\left(\mathcal{E}_i^{\ i}\right)^2 \;+\; \tfrac{1}{2}(\beta+\gamma)\mathcal{E}^2 - \tfrac{1}{2}\gamma\mathcal{J}^2 + \tfrac{1}{2}\beta\mathcal{D}^2 - \left(\alpha+\tfrac{1}{2}(d+1)\beta+\tfrac{1}{2}\gamma\right)\mathcal{D}\,, \end{aligned} \tag{21}$$

where, for $E^i{}_j$ denoting the basic matrices with the matrix elements $(E^i{}_j)^k{}_l = \delta^{ik}\delta_{jl}$,

$$(\mathcal{E}_i{}^j f)(W) = \tfrac{d}{ds}|_{s=0} f\big(\mathrm{e}^{-s\,E^i{}_j} W\big) = -\sum_k W^j{}_k \partial_{W^i{}_k} f(W)\,, \quad \mathcal{E}^2 = \sum_{i,j} \mathcal{E}_i{}^j \mathcal{E}_j{}^i\,,$$
$$\mathcal{J}^2 = -\tfrac{1}{2}\sum_{i,j}\big(\mathcal{E}_i{}^j - \mathcal{E}_j{}^i\big)^2\,, \qquad \mathcal{D} \,=\, \tfrac{d}{ds}|_{s=0} f(\mathrm{e}^s W) = -\sum_i \mathcal{E}_i{}^i\,.$$

Note that $\mathcal{E}^2$ is the quadratic Casimir of $gl(d)$, $\mathcal{J}^2$ the one of $so(d)$ and $\mathcal{D}$ the generator of the dilations. Formula (21) goes back to Ref. [28] where it was discussed for the isotropic incompressible case. For any values of the parameters, the generator $\mathcal{L}$ commutes with the right regular action

$$(R_M f)(W) \,=\, f(WM)$$

of $GL(d)$ on functions on itself.

3.5. Stretching Exponents in the Isotropic Case

The isotropic case with $\alpha = 0$ has been treated in Refs. 2, 3, see also Refs. 7, 28. Here $\mathcal{L}$ commutes also with the left action of $O(d)$ given by

$$(L_O f)(W) \,=\, f(O^{-1}W)\,.$$

In particular, $\mathcal{L}$ preserves the space of functions invariant under the left and right action of $O(d)$, i.e. functions $f(\boldsymbol{\rho})$ that depend on W only through the stretching exponents, see Eq. (8). In other words, the stretching exponents evolve independently of the angles of the $O(d)$ matrices in the decomposition (8). On functions $f(\boldsymbol{\rho})$, the generator $\mathcal{L}$ reduces to the operator

$$\mathcal{L}_{\boldsymbol{\rho}} \,=\, \tfrac{\beta+\gamma}{2}\left(\sum_{i=1}^{d}\frac{\partial^2}{\partial\rho_i^2} \,+\, \sum_{i\neq j}\coth(\rho_i-\rho_j)\,\frac{\partial}{\partial\rho_i}\right)$$
$$+\,\tfrac{\beta}{2}\Big(\sum_{i=1}^{d}\frac{\partial}{\partial\rho_i}\Big)^2 \,-\, \tfrac{(d+1)\beta+\gamma}{2}\sum_{i=1}^{d}\frac{\partial}{\partial\rho_i}\,.$$

The right hand side is the generator of the diffusion process $\boldsymbol{\rho}_t$ satisfying the stochastic differential equations[3]

$$d\rho_i \,=\, \frac{\beta+\gamma}{2}\sum_{j\neq i}\coth(\rho_i-\rho_j)dt \,-\, \tfrac{(d+1)\beta+\gamma}{2}dt \,+\, \zeta_{i\,t}\,dt \tag{22}$$

where $\boldsymbol{\zeta}_t$ is the white noise with the covariance

$$\langle\zeta_{i\,t}\,\zeta_{j\,t'}\rangle \,=\, [(\beta+\gamma)\delta_{ij} \,+\, \beta]\,\delta(t-t')\,.$$

The time t PDF $P_t(\boldsymbol{\rho})$ may be expressed in terms of the heat kernel of the Calogero-Sutherland Hamiltonian.[7,19] The Lyapunov exponents are given by Refs. 4, 25.

$$\lambda_i \;=\; \tfrac{\beta+\gamma}{2}(d-2i+1)-\tfrac{(d+1)\beta+\gamma}{2}$$

and are all different. The large deviation form of $P_t(\boldsymbol{\rho})$ is easy to obtain by the following heuristic considerations.[3] Since $|\rho_i-\rho_j|$ for $i\neq j$ grows approximately linearly with time, at long times $\coth(\rho_i-\rho_j)\approx\pm1$ and the operator $\mathcal{L}_{\boldsymbol{\rho}}$ should reduce to the asymptotic form

$$\mathcal{L}^{as}_{\boldsymbol{\rho}} \;=\; \tfrac{\beta+\gamma}{2}\sum_{i=1}^{d}\frac{\partial^2}{\partial\rho_i^2}+\tfrac{\beta}{2}\left(\sum_{i=1}^{d}\frac{\partial}{\partial\rho_i}\right)^2+\sum_{i=1}^{d}\lambda_i\frac{\partial}{\partial\rho_i}\,.$$

Similarly, the stochastic equation (22) simplifies at long times to

$$d\rho_i \;=\; \lambda_i\,dt\;+\;\zeta_{i\,t}\,dt$$

The long-time asymptotics of ρ_i is now easy to find leading to the large deviation form (10) of the PDF $P_t(\boldsymbol{\rho})$ with the quadratic rate function[3,2]

$$H(\boldsymbol{\rho}/t) \;=\; \tfrac{1}{2(\beta+\gamma)}\left[\sum_{i=1}^{d}\left(\tfrac{\rho_i}{t}-\lambda_i\right)^2-\tfrac{\beta}{(d+1)\beta+\gamma}\left(\sum_{i=1}^{d}\left(\tfrac{\rho_i}{t}-\lambda_i\right)\right)^2\right] \tag{23}$$

taking its minimal value at $\boldsymbol{\rho}/t=\boldsymbol{\lambda}$ and possessing the symmetry (11).

Let us turn now to the stretching exponents $\boldsymbol{\eta}^{\omega}_t(\boldsymbol{r},F)$ along flags $F=OF^0$. Since η_i are defined by the Iwasawa decomposition of the matrix $\tilde{W}=W^{\omega}_t(\boldsymbol{r})\,O$, functions of $\boldsymbol{\eta}$ may be identified with functions f of $\tilde{W}$ invariant under the right action of upper-triangular matrices N with units on the diagonal and the left action by orthogonal matrices O (such functions are necessarily invariant under the right action by diagonal matrices with entries ±1). The calculation of the average of $f(\tilde{W}))$ with respect to the measure $\Sigma(d\boldsymbol{r},dF,d\omega)$ is now simplified by Eq. (17). For fixed $\boldsymbol{r}_0$ and $O\in O(d)$, the statistics of $W^{\omega}_t(\boldsymbol{r}_0)\,O$ may be found by solving the Itô stochastic Eq. (18) with the initial condition $W_0=O$. It follows that

$$\frac{d}{dt}\big\langle f\big(W^{\omega}_t(\boldsymbol{r}_0)\,O\big)\big\rangle \;=\; \big\langle(\mathcal{L}f)\big(W^{\omega}_t(\boldsymbol{r}_0)\,O\big)\big\rangle\,.$$

In the isotropic case with $\alpha=0$, the generator $\mathcal{L}$ preserves the space of function invariant under the right action by upper-triangular matrices N with units on the diagonal and under the left action of orthogonal matrices O and reduces on such functions to the operator

$$\mathcal{L}_{\eta} = \tfrac{\beta+\gamma}{2} \sum_{i=1}^{d} \frac{\partial^2}{\partial \eta_i^2} + \tfrac{\beta}{2} \left(\sum_{i=1}^{d} \frac{\partial}{\partial \eta_i} \right)^2 + \sum_{i=1}^{d} \lambda_i \frac{\partial}{\partial \eta_i}$$

of the form coinciding for all times with the asymptotic form $\mathcal{L}_{\rho}^{as}$ of $\mathcal{L}_{\rho}$. As the result,

$$\left\langle f\left(W_t^{\omega}(\boldsymbol{r}_0)\, O\right)\right\rangle = \int \mathrm{e}^{-t\mathcal{L}_{\eta}}(\mathbf{0}, \boldsymbol{\eta})\, f(\boldsymbol{\eta})\, d\boldsymbol{\eta} = \frac{\int f(\boldsymbol{\eta})\, \mathrm{e}^{-t\, H\left(\eta^1/t,\ldots,\eta^d/t\right)}\, d\boldsymbol{\eta}}{\int \mathrm{e}^{-t\, H\left(\eta^1/t,\ldots,\eta^d/t\right)}\, d\boldsymbol{\eta}}$$

with H given by Eq. (23). In particular, the O-dependence drops out and the integral over the flag variety Fl on the right hand side of Eq. (17) becomes trivial (in fact, in the isotropic case, $\chi(dF) = dF$ i.e. it is $O(d)$-invariant measure on Fl). We infer that the time t PDF $Q_t(\boldsymbol{\eta})$ of the stretching exponents along the (unstable) flags is Gaussian for all times:

$$Q_t(\boldsymbol{\eta}) = \frac{\mathrm{e}^{-t\, H(\eta^1/t,\ldots,\eta^d/t)}}{\int \mathrm{e}^{-t\, H(\eta^1/t,\ldots,\eta^d/t)}\, d\boldsymbol{\eta}}.$$

In particular, its large deviation form coincides with that for the stretching exponents ρ_i except that in the latter case, it is restricted to the region where $\rho_1 > \cdots > \rho_d$. We shall see below that such coincidence of the large deviation statistics for $\boldsymbol{\rho}$ and $\boldsymbol{\eta}$ holds in more general situations.

4. KRAICHNAN FLOW IN A PERIODIC SQUARE

4.1. Generator of the $2d$ Tangent Process

We shall discuss here the statistics of the solutions of the multiplicative stochastic Eq. (18) for 2×2 matrices with the covariance of the white noise strain S_t^{ω} given by Eqs. (19) and (20) with $d = 2$. The overall time scale of the strain is set by

$$\frac{4\delta(0)}{\langle \mathrm{tr}\, S^{\omega T} S^{\omega} \rangle} = \frac{1}{\alpha + \beta + \gamma} \equiv \tau\,.$$

The distribution of S_t^{ω} is invariant under the 90° rotations and reflections in the coordinate axis. If $O_{\frac{\pi}{4}}$ is the rotation by 45°, i.e.

$$O_{\frac{\pi}{4}} = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

then $S_t'^{\omega} = O_{\frac{\pi}{4}} S_t^{\omega} O_{\frac{\pi}{4}}^T$ has the similar distribution to that of S_t^{ω} but with parameters

$$\alpha' = -\alpha\,, \qquad \beta' = \alpha + \beta\,, \qquad \gamma' = \alpha + \gamma$$

Note that $\tau' = \tau,\ 2\gamma' + \alpha' = 2\gamma + \alpha$ and that the compressibility degree is the same for both sets of the parameters. We shall call the ratio

$$\kappa \;=\; |\alpha|\tau \;=\; |\alpha'|\tau'$$

the **anisotropy degree**. It measures the difference between the covariances of the the processes S_t^{ω} and $S_t'^{\omega}$ relative to $\langle \mathrm{tr}\, S^{\omega T} S^{\omega}\rangle$ and it is contained between zero and one. We shall use the dependence on κ to measure the influence of the anisotropy on the distribution of the tangent process W_t. For W_t solving Eq. (18), the distribution of $W_t' = O_{\frac{\pi}{4}} W_t O_{\frac{\pi}{4}}^T$ will coincide with that of the solution of Eq. (18) for the primed values of the parameters. As the result, the distribution of the stretching exponents $\boldsymbol{\rho}_t^{\omega}(\boldsymbol{r})$ as well as that of $\boldsymbol{\eta}_t^{\omega}(\boldsymbol{r})$ for the two sets of parameters are identical. Below, we shall then restrict ourselves to the case with $\alpha \geq 0$.

The parametrization (8) takes in two dimensions the form:

$$W \;=\; \begin{pmatrix} \cos\frac{\phi}{2} & \sin\frac{\phi}{2} \\ -\sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{pmatrix} \begin{pmatrix} \mathrm{e}^{\rho_1} & 0 \\ 0 & \mathrm{e}^{\rho_2} \end{pmatrix} \begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix}$$

(we may assume that $\det W = 1$). Hence functions of W may be viewed as functions of two angles and the stretching exponents and satisfying the relations

$$\begin{aligned} f(\phi, \rho_1, \rho_2, \psi) \;&=\; f\left(\phi + \pi, \rho_2, \rho_1, \psi - \tfrac{\pi}{2}\right)\,, \\ f(\phi, \rho_1, \rho_2, \psi) \;&=\; f\left(\phi, \rho_1, \rho_2, \psi + 2\pi\right) \end{aligned}$$

that permit to restrict the parameters to the region $\rho_1 \geq \rho_2$ and $0 \leq \phi, \psi \leq 2\pi$. For the generator of the tangent process given by Eq. (21) one obtains the following complicated expression:

$$\begin{aligned} \mathcal{L} = {} & \tfrac{1}{2}\alpha[\partial_{\rho_1} + \partial_{\rho_2}]^2 \\ & + \tfrac{1}{2}\alpha[2\,\sin\phi\,\coth(\rho_1 - \rho_2)\,\partial_\phi \,-\, \cos\phi\,(\partial_{\rho_1} - \partial_{\rho_2}) \,-\, \sin\phi\,\sinh^{-1}(\rho_1 - \rho_2)\,\partial_\psi]^2 \\ & + \tfrac{1}{2}(\beta + \gamma)[\partial_{\rho_1}^2 + \partial_{\rho_2}^2 + 2\,\sinh^{-2}(\rho_1 - \rho_2)\,\partial_\phi^2 + \tfrac{1}{2}\,\sinh^{-2}(\rho_1 - \rho_2)\,\partial_\psi^2 \\ & - 2\,\cosh(\rho_1 - \rho_2)\,\sinh^{-2}(\rho_1 - \rho_2)\,\partial_\phi\partial_\psi + \coth(\rho_1 - \rho_2)\,(\partial_{\rho_1} - \partial_{\rho_2})] \\ & + 2\,\gamma\,\partial_\phi^2 + \tfrac{1}{2}\beta[\partial_{\rho_1} + \partial_{\rho_2}]^2 - \tfrac{1}{2}(2\alpha + 3\beta + \gamma)\,[\partial_{\rho_1} + \partial_{\rho_2}]\,. \end{aligned} \tag{24}$$

$\mathcal{L}$ is self-adjoint in the L^2 scalar product with the measure $\mathrm{e}^{-\rho_1-\rho_2}\sinh|\rho_1 - \rho_2|\,d\phi\, d\rho_1 d\rho_2 d\psi$. Note that $\mathcal{L}$ commutes separately with the translation of ϕ by π, with permutation of ρ_i and with the arbitrary translations of ψ. Since at the end we shall be interested in the distribution of the (ordered) stretching exponents, we may right away restrict ourselves to the sector of functions that are periodic in

ϕ of period π, even under the interchange of ρ^i and independent of ψ. Upon setting

$$\tfrac{1}{2}(\rho_1+\rho_2)\equiv r\,,\qquad \rho_1-\rho_2\equiv\rho$$

$\mathcal{L}$ reduces in the action on ψ-independent functions to the operator $\mathcal{L}_r+\mathcal{L}_{\phi\rho}$ with

$$\mathcal{L}_r = \tfrac{1}{4}(2\alpha+3\beta+\gamma)\,\partial_r^2 - \tfrac{1}{2}(2\alpha+3\beta+\gamma)\,\partial_r\,, \tag{25}$$

$$\begin{aligned}\mathcal{L}_{\phi\rho} &= 2\alpha\big[\sin\phi\,\coth\rho\,\partial_\phi - \cos\phi\,\partial_\rho\big]^2 + 2\gamma\,\partial_\phi^2\\ &\quad+\tfrac{1}{2}(\beta+\gamma)\big[2\,\partial_\rho^2 + 2\,\sinh^{-2}\rho\,\partial_\phi^2 + 2\,\coth\rho\,\partial_\rho\big].\end{aligned} \tag{26}$$

It follows that at all times the processes r_t and (ρ_t,ϕ_t), starting at $r_0=0=\rho_0$ and $\phi_0=0$, are independent.

4.2. Sum of Lyapunov Exponents

The PDF of r_t takes for all times the Gaussian large deviation form since $\mathcal{L}_r$ has constant coefficients:

$$P_t(r) = \frac{1}{\sqrt{\pi(2\alpha+3\beta+\gamma)t}}\,\mathrm{e}^{-t\,H_r(r/t)} \tag{27}$$

with the quadratic rate function

$$H_r(r/t) = \frac{\big(r/t+\frac{1}{2}(2\alpha+3\beta+\gamma)\big)^2}{2\alpha+3\beta+\gamma}\,.$$

In particular, when $t\to\infty$, the PDF of r concentrates at $\frac{r}{t}=-\frac{1}{2}(2\alpha+3\beta+\gamma)$ which gives the half of the sum of the Lyapunov exponents:

$$\lambda_1+\lambda_2 = -(2\alpha+3\beta+\gamma)\,. \tag{28}$$

Note that if one normalizes the Lyapunov exponents by multiplying them by the overall time scale $\tau=(\alpha+\beta+\gamma)^{-1}$ then one obtains the relation

$$(\lambda_1+\lambda_2)\tau = -2\wp \tag{29}$$

stating that the normalized sum of the Lyapunov exponents is directly tied to the compressibility degree.

The Evans-Cohen-Morriss-Gallavotti-Cohen symmetry (11) involves here only the large deviations of r_t and reduces to the identity

$$H_r(-r/t) = H_r(r/t) - 2r/t\,. \tag{30}$$

4.3. Asymptotic Form of the (ρ, ϕ)-Process

It remains to find the large-time form of the joint PDF of ϕ and ρ. Unlike in the isotropic situation, the evolution of ρ does not decouple from the angle ϕ and both have to be treated at the same time. Recall that we may restrict ourselves to the sector of functions of period π in ϕ and even in ρ so that we may consider only $\rho \geq 0$ (i.e. $\rho_1 \geq \rho_2$) imposing the appropriate boundary condition at $\rho = 0$ (recall that $\mathcal{L}$ acts at smooth functions on $GL(d)$). At long times, ρ/t will still concentrate at a single value equal to the difference of the Lyapunov exponents. Anticipating the latter to be strictly positive (except in the case when $\beta = \gamma = 0$ treated in Sect. 4.I), we infer that at large t the process ρ_t will take predominantly large values $\propto t$. Consequently, we should be able to replace $\coth\rho$ by 1 in the expression (26) for the generator $\mathcal{L}_{\phi\rho}$ and drop the term with $\sinh^{-2}\rho$ reducing $\mathcal{L}_{\phi\rho}$ to the asymptotic form:

$$\mathcal{L}^{as}_{\phi\rho} \;=\; 2\alpha\big[\sin\phi\,\partial_\phi \;-\; \cos\phi\,\partial_\rho\big]^2 \;+\; 2\gamma\,\partial_\phi^2 \;+\; (\beta+\gamma)\big[\partial_\rho^2 \;+\; \partial_\rho\big]. \tag{31}$$

Note on the margin that under a similar transformation applied to operator $\mathcal{L}$ of Eq. (24), all terms involving the angle ψ drop out implying that ψ becomes frozen at long times, in agreement with the Oseledec theorem. With an appropriate boundary conditions at $\rho = 0$, the operator $\mathcal{L}^{as}_{\phi\rho}$ is self-adjoint with respect to the L^2 scalar product with the measure $\mathrm{e}^{\rho}d\rho\, d\phi$. The restriction to $\rho \geq 0$ will not, however, effect the large deviation form of the PDF of ρ. In what follows, we shall then simplify the things considering the operator $\mathcal{L}^{as}_{\phi\rho}$ as acting on functions of ρ defined on the whole line, restricting ρ to positive values only at the very end.

Although $\mathcal{L}^{as}_{\phi\rho}$ still does not preserve the subspace of functions that depend only on ρ, it does preserve the one of functions that depend only on ϕ. Hence the evolution of ϕ becomes at long times independent of that of ρ although the opposite is not true. More specifically, the angle ϕ undergoes at long time a diffusion on the circle of circumference π whose generator

$$\mathcal{L}^{as}_{\phi} \;=\; 2\alpha\big[\sin\phi\,\partial_\phi\big]^2 \;+\; 2\gamma\,\partial_\phi^2 \tag{32}$$

is obtained by restricting $\mathcal{L}^{as}_{\phi\rho}$ to functions independent of ρ. Such diffusion process converges exponentially fast (we shall find the rate of the exponential convergence below) to a stationary state. The stationary density

$$\chi(\phi) \;=\; \frac{\big[\gamma + \alpha\,\sin^2\phi\big]^{-1/2}}{\int_0^\pi \big[\gamma + \alpha\,\sin^2\varphi\big]^{-1/2} d\varphi}\,. \tag{33}$$

is the unique positive normalized solution of period π of the equation

$$\mathcal{L}^{as\,\dagger}_{\phi}\,\chi(\phi) \;=\; \big(2\alpha[\partial_\phi\,\sin\phi]^2 \;+\; 2\gamma\,\partial_\phi^2\big)\,\chi(\phi) \;=\; 0\,.$$

4.4. Difference of Lyapunov Exponents

The difference of the Lyapunov exponents may be found now by applying the following strategy. Suppose that there exists a smooth periodic function $f(\phi)$ of period π such that

$$\mathcal{L}^{as}_{\phi\rho}\,(\rho + f(\phi)) \;=\; \lambda \;=\; \text{const.} \tag{34}$$

It follows that

$$\frac{d}{dt}\langle\rho\rangle \;=\; \lambda \,-\, \frac{d}{dt}\langle f(\phi)\rangle \;\underset{t\to\infty}{\longrightarrow}\; \lambda$$

with the exponentially fast convergence. We infer then that λ is equal to the mean asymptotic rate of change of ρ that, in turn, is equal to the difference of the Lyapunov exponents. Identity (34) may be rewritten in somewhat more explicit form as

$$2\alpha\,\sin^2\phi \,+\, \beta \,+\, \gamma \,+\, \mathcal{L}^{as}_{\phi}\, f(\phi) \;=\; \lambda\,.$$

Integrating the latter equality against $\chi(\phi)$, we obtain the relation

$$\lambda \;=\; \int_0^{\pi}\,\big[\beta \,+\, \gamma \,+\, 2\alpha\,\sin^2\varphi\big]\,\chi(\varphi)\,d\phi \tag{35}$$

that fixes the value of λ. It is easy to see that Eq. (35) is also a sufficient condition for the existence of the function $f(\phi)$ satisfying condition (34). Substituting the explicit expression (33) for $\chi(\phi)$, we infer that

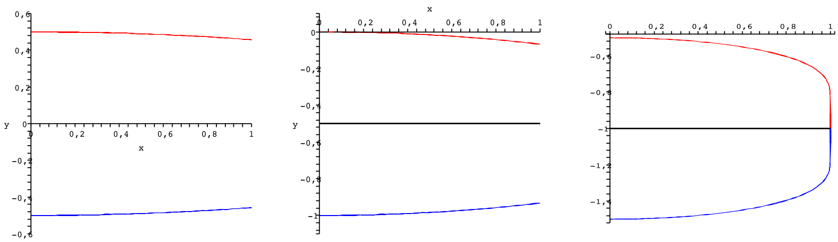
$$\lambda_1 - \lambda_2 \;=\; \beta \,-\, \gamma \,+\, 2\,\frac{\int_0^{\pi}[\gamma \,+\, \alpha\,\sin^2\varphi]^{1/2}\,d\varphi}{\int_0^{\pi}[\gamma \,+\, \alpha\,\sin^2\varphi]^{-1/2}\,d\varphi}\,. \tag{36}$$

Recall that $\gamma \geq |\beta|$ and $2\alpha + 2\beta + \gamma \geq |\beta|$ and that we have assumed that $\alpha \geq 0$. It follows that $\lambda^1 - \lambda^2 \geq \beta + \gamma \geq 0$ and at least one of the last inequalities is sharp unless $\beta = \gamma = 0$. Hence $\lambda_1 > \lambda_2$ if $\beta + \gamma > 0$ which is consistent with the assumption that, typically, ρ becomes large for long times. The integrals are given by the elliptic functions $\boldsymbol{K}(k)$ and $\boldsymbol{E}(k)$ with the modulus $k = \sqrt{\frac{\alpha}{\alpha+\gamma}}$:

$$\frac{\int_0^{\pi}[\gamma \,+\, \alpha\,\sin^2\varphi]^{1/2}\,d\varphi}{\int_0^{\pi}[\gamma \,+\, \alpha\,\sin^2\varphi]^{-1/2}\,d\varphi} \;=\; (\alpha+\gamma)\,\frac{\boldsymbol{E}(k)}{\boldsymbol{K}(k)}\,.$$

Fig. 1 depicts the normalized Lyapunov exponents $\lambda_i\tau$ for $\tau = (\alpha + \beta + \gamma)^{-1}$ as functions of the anisotropy degree $\kappa = |\alpha|\tau$,

$$\lambda_1\tau \;=\; -\,1 \,+\, \frac{3 - 2\wp + \kappa}{2}\,\frac{\boldsymbol{E}\left(\sqrt{\frac{2\kappa}{3-2\wp+\kappa}}\right)}{\boldsymbol{K}\left(\sqrt{\frac{2\kappa}{3-2\wp+\kappa}}\right)}\,,$$



(a) $\wp = 0$ (b) $\wp = \frac{1}{2}$ (c) $\wp = 1$

Fig. 1. (Color online) Normalized Lyapunov exponents $\lambda_1\tau$ and $\lambda_2\tau$ as functions of anisotropy degree $\kappa = |\alpha|\tau$ for three values of the compressibility degree $\wp$.

$$\lambda_2\tau \;=\; 1 \;-\; 2\wp \;-\; \frac{3-2\wp+\kappa}{2}\,\frac{\boldsymbol{E}\left(\sqrt{\frac{2\kappa}{3-2\wp+\kappa}}\right)}{\boldsymbol{K}\left(\sqrt{\frac{2\kappa}{3-2\wp+\kappa}}\right)}\,,$$

for three values of the compressibility degree $\wp = 0$, $\wp = \frac{1}{2}$ and $\wp = 1$. Note that $\lambda_1\tau$ decreases and $\lambda_2\tau$ increases with κ at constant compressibility degree $\wp$, with the sum of the two fixed to $-2\wp$, see Eq. (29). The incompressible system stays always chaotic (i.e. with positive top Lyapunov exponent) and this is also true for sufficiently small compressibility degree. For $\wp$ slightly below $\frac{1}{2}$, however, an increase of κ may kill chaos. For $\wp \geq \frac{1}{2}$ the system is never chaotic. For $\wp = 1$, the tendency of anisotropy to bring the Lyapunov exponents closer attains its maximum with the two Lyapunov exponents coinciding for the extreme anisotropy when S_t^ω is a diagonal matrix with independent equally distributed entries representing independent stretching and contraction along the coordinate axes (or when it is the 45^o rotation of such a matrix).

4.5. Large Deviations for Exponents ρ

The joint PDF of ϕ_t and ρ_t takes the form of the heat kernel

$$P_t(\phi,\rho) \;=\; \mathrm{e}^{t\,\mathcal{L}_{\phi\rho}}(0,0;\phi,\rho)$$

and for large time t should be well approximated by the modified heat kernel

$$P_t^{as}(\phi,\rho) \;=\; \mathrm{e}^{t\,\mathcal{L}_{\phi\rho}^{as}}(0,0;\phi,\rho)\,.$$

In the latter, the ρ-contribution may be diagonalized by the Fourier transform so that we get

$$P_t^{as}(\phi,\rho) \;=\; \int_{\mathcal{C}} \mathrm{e}^{-\nu\rho\,+\,t\,(\beta+\gamma)\nu(\nu+1)}\;\mathrm{e}^{t\,\mathcal{L}_\nu}(0,\phi)\;\frac{d\nu}{2\pi i} \tag{37}$$

where the integration is over a line $Re\,\nu = -1/2$ parallel to the imaginary axis and

$$\mathcal{L}_\nu \;=\; 2\alpha\big[\sin\phi\,\partial_\phi \;-\; \nu\,\cos\phi\big]^2 \;+\; 2\gamma\,\partial_\phi^2$$

is a second order differential operator acting on periodic functions of period π. For $Re\,\nu = -1/2$, $\mathcal{L}_\nu$ is self-adjoint with respect to the L^2 scalar product with the measure $d\phi$. As we shall see below, the contour $\mathcal{C}$ of integration in Eq. (37) may be moved to any line parallel to the imaginary axis. Operator $\mathcal{L}_\nu$ may be viewed as a perturbation of the generator $\mathcal{L}_\phi^{as}$ of Eq. (32) with which it coincides for $\nu = 0$. By a rescaling, a similarity transform and the elliptic change of variables

$$\phi \;\longmapsto\; u(\phi) \;=\; \int_0^\phi \Big[\frac{\alpha+\gamma}{\gamma \,+\, \alpha\,\sin^2\psi}\Big]^{1/2} d\psi\,,$$

$\mathcal{L}_\nu$ is put into the form of a one-dimensional Schrödinger operator:

$$-\,\tfrac{1}{2(\alpha+\gamma)}\,\mathrm{e}^{-\nu\,h(\phi)}\,\mathcal{L}_\nu\,\mathrm{e}^{\nu\,h(\phi)} \;=\; -\frac{d^2}{du^2} \;+\; \nu(\nu+1)\,V(\phi(u)) \tag{38}$$

for the function

$$h(\phi) \;=\; \tfrac{1}{2}\,\ln\big[\gamma \,+\, \alpha\,\sin^2\phi\big]$$

and the attractive potential depicted on Fig. 2,

$$V(\phi(u)) \;=\; \frac{\gamma}{\alpha+\gamma} \;-\; \frac{\gamma}{\gamma \,+\, \alpha\,\sin^2\phi} \;=\; -k^2\,\mathrm{cn}^2(u,k) \;=\; -k^2 \,+\, k^2\,\mathrm{sn}^2(u,k)\,,$$

where $\mathrm{sn}(u,k)$ and $\mathrm{cn}(u,k)$ are the Jacobi elliptic function corresponding to the modulus $k = \sqrt{\frac{\alpha}{\alpha+\gamma}}$. The Schrödinger operator of Eq. (38) acts on periodic functions of u with the period $2\boldsymbol{K}(k)$ (that corresponds to the period π in ϕ). Up to a constant, it is equal to the Lamé integrable operator in the Jacobian form, see Refs. 14, 30,

$$\mathcal{H}_\nu \;=\; -\frac{d^2}{du^2} \;+\; \nu(\nu+1)\,k^2\,\mathrm{sn}^2(u,k)\,.$$

We thus obtain the relation

$$P_t^{as}(\phi,\rho) \;=\; \int_{\mathcal{C}} \mathrm{e}^{-\nu\rho\,+\,t\,(2\alpha+\beta+\gamma)\nu(\nu+1)\,+\,\nu(h(0)-h(\phi))}\;\mathrm{e}^{-2(\alpha+\gamma)t\,\mathcal{H}_\nu}(0,u(\phi))\;\frac{du(\phi)}{d\phi}\,\frac{d\nu}{2\pi i} \tag{39}$$

Note that by the Feynman-Kac formula, for $\nu = \nu_1 + i\,\nu_2$ with $\nu_{1,2}$ real, the absolute value of the integrand on the right hand side is bounded by

$$\mathrm{e}^{-\nu_1\rho\,+\,t\,(\beta+\gamma)\big[\nu_1(\nu_1+1)-\nu_2^2\big]\,+\,\nu_1(h(0)-h(\phi))}\;\mathrm{e}^{-2(\alpha+\gamma)t\left(-\frac{d^2}{du^2}+[\nu_1(\nu_1+1)-\nu_2^2]V(\phi(u))\right)}(0,u(\phi))\;\frac{du(\phi)}{d\phi}\,.$$

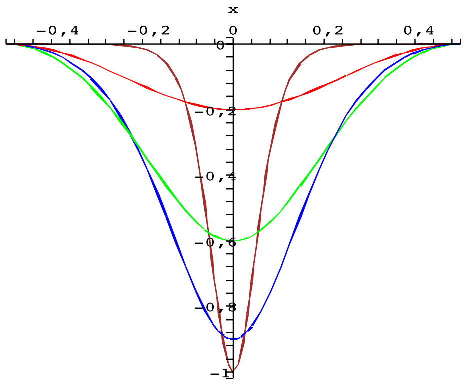


Fig. 2. (Color online) $V(\phi(u))$ as a function of $\frac{u}{2\boldsymbol{K}(k)}$ for, from middle top to bottom for $u = 0$, $k^2 = \frac{\alpha}{\alpha+\gamma} = 0.2,\ 0.6,\ 0.9$ and 0.99999.

For $\gamma > 0$, the last expression tends to zero when $\nu_2 \to \pm\infty$ (uniformly on bounded intervals of ν_1) since $\beta + \gamma \geq 0$ and V is attractive. It follows then that the contour of integration $\mathcal{C}$ on the right hand side of Eq. (39) may be, as announced, moved to any line parallel to the imaginary axis. Let us consider the spectral decomposition

$$\mathrm{e}^{-2(\alpha+\gamma)t\,\mathcal{H}_\nu} \;=\; \sum_{n=0}^{\infty} \mathrm{e}^{-2(\alpha+\gamma)t\,E_{\nu,n}}\,|\Psi_{\nu,n}\rangle\langle\Psi_{\nu,n}| \tag{40}$$

with the eigenvalues $E_{\nu,n} = E_{\nu,n}(k^2)$ of $\mathcal{H}_\nu$ ordered in a non-decreasing way for ν real. For large t, the dominant contribution comes from the ground state $\Psi_{\nu,0}$ of $\mathcal{H}_\nu$. In particular, for the vanishing coupling constant, $\Psi_{0,0} = (2\boldsymbol{K}(k))^{-1/2}$, $E_{0,0} = 0$, $E_{0,1} = E_{0,2} = \pi^2\boldsymbol{K}(k)^{-2}$ and $\mathrm{e}^{t\,\mathcal{H}_0}(0; u(\phi))$ converges at long times to $(2\boldsymbol{K}(k))^{-1}$ with the exponential rate equal to $2\pi^2(\alpha+\gamma)\boldsymbol{K}(k)^{-2}$. Note that this rate goes to zero when γ tends to zero since the half-period $\boldsymbol{K}(k)$ diverges in this limit.

Insertion of the expansion (40) into the expression (39) permits to extract the large deviation form of the PDF of ρ_t from the ground state contribution:

$$\int_0^{\pi} P_t^{as}(\phi,\rho)\,d\phi \;\propto\; \int_{\mathcal{C}} \mathrm{e}^{-t\left[\nu\rho/t\,-\,(2\alpha+\beta+\gamma)\nu(\nu+1)\,+\,2(\alpha+\gamma)\,E_{\nu,0}(k^2)\right]}\,\frac{d\nu}{2\pi i} \;\propto\; \mathrm{e}^{-t\,H_\rho(\rho/t)} \tag{41}$$

with the rate function

$$H_\rho(\rho/t) \;=\; \max_{\nu}\left[\nu\rho/t \;-\; (2\alpha+\beta+\gamma)\nu(\nu+1) \;+\; 2(\alpha+\gamma)\,E_{\nu,0}(k^2)\right] \tag{42}$$

defined for $\rho/t > 0$. We shall denote by $\nu_{\max}$ the value of ν where the maximum is attained (the maximum over real ν corresponds to a minimum of the real part along the axis parallel to the imaginary one). Note that only the ground state energy of operator $\mathcal{H}_\nu$ contributes to the rate function H_ρ. The ground state wave function enters the prefactors in the long time asymptotics of the PDF of ρ. Let us check that when $t \to \infty$ then ρ/t concentrates at the value equal to the difference of the Lyapunov exponents as given by Eq. (36). To this end, we must find the minimum of $H_\rho(\rho/t)$. The stationarity condition implies the equations

$$\nu = 0\,, \qquad \rho/t \;=\; (2\alpha+\beta+\gamma)(2\nu+1) \;-\; 2(\alpha+\gamma)\,\partial_\nu E_{\nu,0}(k^2) \tag{43}$$

and the minimizing value of ρ/t is given by the relation

$$(\rho/t)_{\min} \;=\; 2\alpha \;+\; \beta \;+\; \gamma \;-\; 2(\alpha+\gamma)\,\partial_\nu E_{\nu,0}(k^2)|_{\nu=0}\,.$$

The derivative of the ground state energy may be calculated by the first order perturbation theory:

$$\begin{aligned}\partial_\nu E_{\nu,0}|_{\nu=0} \;&=\; \langle\Psi_{0,0}|k^2\,\mathrm{sn}^2(\cdot\,,k)|\Psi_{0,0}\rangle \;=\; k^2 \;+\; \frac{1}{2\boldsymbol{K}(k)}\int_0^{2\boldsymbol{K}(k)} V(\phi(u)\,du\\ &=\; 1 \;-\; \frac{1-k^2}{2\boldsymbol{K}(k)}\int_0^{\pi}\left[1-k^2\,\cos^2\phi\right]^{-3/2} d\phi \;=\; 1 \;-\; \frac{\boldsymbol{E}(k)}{\boldsymbol{K}(k)}\end{aligned}$$

where the last equality follows from the elliptic identity 2.584.37 in Ref. (21). We obtain this way the relation

$$(\rho/t)_{\min} \;=\; \beta \;-\; \gamma \;+\; 2(\alpha+\gamma)\,\frac{\boldsymbol{E}(k)}{\boldsymbol{K}(k)}$$

which agrees with Eq. (36) for $\lambda_1 - \lambda_2 \equiv \lambda$.

A closed expression for the second derivative of the rate function H_ρ may be obtained by differentiating twice the defining relation (42):

$$H''_\rho(\rho/t) \;=\; \frac{d\nu_{\max}}{d(\rho/t)} \;=\; \frac{1}{2\left(2\alpha+\beta+\gamma\;-\;(\alpha+\gamma)\,\partial_\nu^2 E_{\nu,0}(k^2)|_{\nu=\nu_{\max}}\right)} \tag{44}$$

since $\nu_{\max}$ is related to ρ/t by the second of Eqs. (43) whose differentiation leads to the last equality. In particular,

$$H''_\rho(\lambda) \;=\; \frac{d\nu_{\max}}{d(\rho/t)}\Big|_\lambda \;=\; \frac{1}{2\left(2\alpha+\beta+\gamma\;-\;(\alpha+\gamma)\,\partial_\nu^2 E_{\nu,0}(k^2)|_{\nu=0}\right)} \tag{45}$$

By the second order perturbation expansion,

$$\begin{aligned}\partial_\nu^2 E_{\nu,0}|_{\nu=0} &= 2\partial_\nu E_{\nu,0}|_{\nu=0} - 2\sum_{n=1}^{\infty} E_{0,n}^{-1}\, |\langle\Psi_{0,0}|k^2\,\mathrm{sn}^2(\cdot,k)|\Psi_{0,n}\rangle|^2 \\ &= 2 - 2\frac{\boldsymbol{E}(k)}{\boldsymbol{K}(k)} - \frac{k^4\boldsymbol{K}(k)^2}{\pi^2}\sum_{n=1}^{\infty} n^{-2} f_n(k)^2 \end{aligned} \tag{46}$$

with the Fourier coefficients $f_n(k) = \int_{-1}^{1} \cos(\pi n x)\,\mathrm{sn}^2(\boldsymbol{K}(k)x, k)\,dx$. $H''_\rho(\lambda)$ is equal to the inverse variance of the normal random variable obtained by the central limit $\lim\limits_{t\to\infty} (\rho_t - \lambda\, t)/\sqrt{t}$.

Combining Eqs. (42) and (27), one obtains the following form of the large deviations rate functions for the stretching exponents:

$$\begin{aligned} H(\rho_1/t, \rho_2/t) &= \frac{(\frac{\rho_1}{t} + \frac{\rho_2}{t} + 2\alpha + 3\beta + \gamma)^2}{4(2\alpha + 3\beta + \gamma)} \\ &\quad + \max_\nu \Big[\nu\big(\tfrac{\rho_1}{t} - \tfrac{\rho_2}{t}\big) - (2\alpha\beta + \gamma)\nu(\nu+1) \\ &\quad + 2(\alpha+\gamma)\, E_{\nu,0}\big(\tfrac{\alpha}{\alpha+\gamma}\big)\Big]. \end{aligned} \tag{47}$$

for $\rho_1 > \rho_2$. When $\alpha = 0$ (the isotropic case) then $E_{\nu,0} = 0$ and the large deviation rate function (47) reduces to the one given by the two-dimensional version of expression (23).

4.6. Large Deviations for Exponents η

Similarly as in the isotropic case, the PDF of the stretching exponents $\boldsymbol{\eta}_t$ along the unstable flags may be obtained from Eq. (17) by considering the Iwasawa decomposition

$$\tilde{W} = \begin{pmatrix} \cos\frac{\phi}{2} & \sin\frac{\phi}{2} \\ -\sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{pmatrix} \begin{pmatrix} e^{\eta_1} & 0 \\ 0 & e^{\eta_2} \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}. \tag{48}$$

of the solutions of the linear stochastic Eq. (18) with the initial condition given by a (random) rotation matrix. In the parametrization (48) and upon the substitution $\frac{1}{2}(\eta_1 + \eta_2) = r$, $\eta_1 - \eta_2 = \eta$, the generator $\mathcal{L}$ of the tangent process takes the form:

$$\begin{aligned}\mathcal{L} = {} & \tfrac{1}{2}(2\alpha + \beta + \gamma)\big[\tfrac{1}{2}\partial_r^2 + 2\cos^2\phi\,\partial_\eta^2 + 2\sin^2\phi\,\partial_\phi^2 + 2\sin^2\phi\,e^{-2\eta}\partial_x^2 \\ & - 2\sin(2\phi)\,\partial_\eta\partial_\phi + 2\sin(2\phi)\,e^{-\eta}\partial_\eta\partial_x - 4\sin^2\phi\,e^{\eta}\partial_\phi\partial_x + 2\sin^2\phi\,\partial_\eta \\ & + \sin(2\phi)\,\partial_\phi - 2\sin(2\phi)\,e^{-\eta}\partial_x\big] + \tfrac{1}{2}(\beta+\gamma)\big[2\sin^2\phi\,\partial_\eta^2 - 2\sin^2\phi\,\partial_\phi^2 \\ & + 2\cos^2\phi\,e^{-2\eta}\partial_x^2 + 2\sin(2\phi)\,\partial_\eta\partial_\phi - 2\sin(2\phi)\,e^{-\eta}\partial_\eta\partial_x - 4\cos^2\phi\,e^{-\eta}\partial_\phi\partial_x \\ & + 2\cos^2\phi\,\partial_\eta - \sin(2\phi)\,\partial_\phi + 2\sin(2\phi)\,e^{-\eta}\partial_x\big] \\ & + 2\gamma\,\partial_\phi^2 + \tfrac{1}{2}\beta\,\partial_r^2 - \tfrac{1}{2}(2\alpha + 3\beta + \gamma)\partial_r\,. \end{aligned}$$

It is self-adjoint in the L^2 scalar product with the measure $\mathrm{e}^{-2r+\eta}d\phi\, dr\, d\eta\, dx$. In the action on functions independent on x, $\mathcal{L}$ reduces to the sum $\mathcal{L}_r + \mathcal{L}_{\phi\eta}$ with $\mathcal{L}_r$ given by Eq. (25) and

$$\mathcal{L}_{\phi\eta} \;=\; 2\alpha\big[\sin\phi\,\partial_\phi \;-\; \cos\phi\,\partial_\eta\big]^2 \;+\; 2\gamma\,\partial_\phi^2 \;+\; (\beta+\gamma)\big[\partial_\eta^2 \;+\; \partial_\eta\big]. \qquad (49)$$

Note that the operator $\mathcal{L}_{\phi\eta}$ has the form identical to the asymptotic form $\mathcal{L}^{as}_{\phi\rho}$ of $\mathcal{L}_{\phi\rho}$, see Eq. (31). In principle, it acts now on functions of ϕ with period 4π but it preserves the subspace of functions with period π. The evolution of r decouples from that of ϕ and η leading to the PDF (27).

As for the joint time t PDF of ϕ and η, it is related to the heat kernel of $\mathcal{L}_{\phi\eta}$ by the equality

$$P_t(\phi,\eta) \;=\; \int_0^\pi \mathrm{e}^{t\,\mathcal{L}_{\phi\eta}}(\phi_0, 0; \phi, \eta)\,\chi(\phi_0)\,d\phi_0 \qquad (50)$$

obtained from Eq. (17). Indeed, it follows from the relation relation (16) that the probability measure $\chi(dF)$ on the flag variety Fl of the 1-dimensional subspaces spanned by vectors $(\cos\frac{\phi_0}{2}, -\sin\frac{\phi_0}{2})$ has to be proportional to $\chi(\phi_0)\,d\phi_0$ where the function $\chi(\phi)$ is the stationary density (33) of the angle ϕ. Since $\chi(\phi)$ is periodic with period π (as a consequence of the invariance of the law of the tangent process with respect to the rotations by 90^o), so must be $P_t(\,\cdot\,,\eta)$ and we could restrict the ϕ_0-integration in Eq. (50) to the interval $[0,\pi[$. The rest of the analysis of the large deviations of η runs as for the large deviations of ρ. As the result, the large deviation rate function for $\boldsymbol{\eta}$ has the same functional form (47) as that for $\boldsymbol{\rho}$ except of the absence of the restriction $\eta_1 > \eta_2$. In particular, the rate function H_η of the difference $\eta = \eta_1 - \eta_2$ of the stretching exponents coincides with the function H_ρ as defined by Eq. (42) on the whole real line rather than on the half line:

$$H_\eta(\eta/t) \;=\; \max_\nu \big[\nu\eta/t \;-\; (2\alpha+\beta+\gamma)\nu(\nu+1) \;+\; 2(\alpha+\gamma)\,E_{\nu,0}(k^2)\big]. \qquad (51)$$

Note that $H_\eta(\eta/t) + \eta/(2t)$ is then an even function of η/t. Indeed, the values of $\nu_{\max}$ at which the maximum on the right hand side of Eq. (51) is attained for η/t and for $-\eta/t$ are related by the reflection around $\nu = -1/2$. They are smaller than -1 for $\eta/t < -\lambda$, lie in between -1 and 0 for $-\lambda < \eta/t < \lambda$ and are positive for $\eta/t > \lambda$. Fig. 3 presents the graph of the maximizing $\nu_{\max}$ as a function of $(\eta/t)\tau$ for $\kappa = 0.8$ and $\wp = 0.9$. We obtain this way the fluctuation relation that compares the rate function for opposite values of η/t:

$$H_\eta(-\eta/t) \;=\; H_\eta(\eta/t) \;+\; \eta/t. \qquad (52)$$

It resembles the Evans-Cohen-Morriss-Gallavotti-Cohen relation (30) but is different from it (recall that r and η are independent random variables at all times).

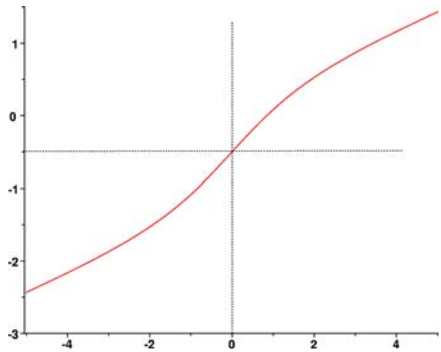


Fig. 3. (Color online) $\nu_{\max}$ as a function of $(\eta/t)\tau$ for $\kappa = 0.8$ and $\wp = 0.9$.

Although η/t may take negative values, Eq. (52) implies that the probability of such events is exponentially suppressed for large t when $\lambda_1 > \lambda_2$.

4.7. Properties of the Rate Function H_η

The large deviations rate function H_η is related to the ground state energy $E_{\nu,0}(k^2)$ of the Lamé operator $\mathcal{H}_\nu$ for $k^2 \equiv \frac{\alpha}{\alpha+\gamma}$ by the formula (51). A lot is known about the eigenvalues and eigenfunctions of $\mathcal{H}_\nu$. The power series expansions for the eigenfunctions in an appropriate parametrization may be obtained by solving a recursion relation[30] or by diagonalizing tridiagonal matrices.[14] For ν a positive integer and the lowest $2\nu + 1$ eigenvalues, the corresponding matrices become finite and one obtains as the eigenfunctions the "Lamé polynomials". The Lamé operator may be diagonalized (for general quasi-momenta) by the Bethe Ansatz[13,20] that in this case goes back to the work of Hermite[30] in the second half of the nineteenth century. A simple Maple program computes $E_{\nu,0}(k^2)$.[29] We also used a C program to compute $E_{\nu,0}$ by solving directly the eigenvalue equation. Fig. 4 presents the graph of $E_{\nu,0}(k^2)$ as a function of ν for $k^2 = 0.2,\ 0.4,\ 0.6,\ 0.8$ and 1. For large values of ν,[29]

$$E_{\nu,0}(k^2) = |k\nu| - \tfrac{1}{4}(1 + k^2) + \mathcal{O}(|\nu|^{-1}).$$

This leads to the following large $|\rho|/t$ behavior of H_η:

$$H_\eta(\eta/t) = -\frac{\eta}{2t} + \frac{\left(\frac{|\eta|}{2t} + \sqrt{\alpha(\alpha+\gamma)}\right)^2}{2\alpha + \beta + \gamma} - \frac{2\alpha + \gamma}{4} + \mathcal{O}(|\eta|^{-1}).$$

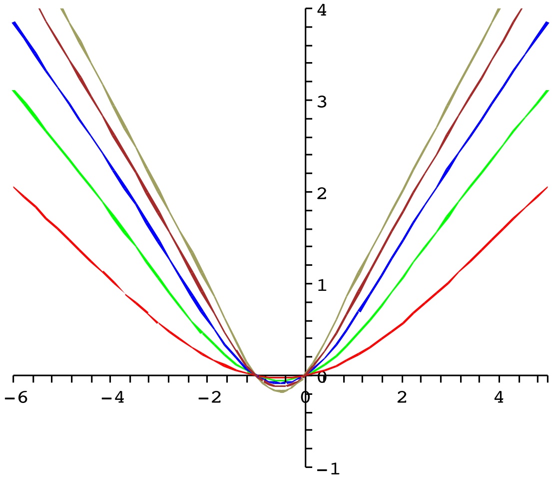


Fig. 4. (Color online) $E_{\nu,0}(k^2)$ as a function of ν from bottom to top for $k^2 = 0.2,\ 0.4,\ 0.6,\ 0.8$ and 1.

Note that the left and right asymptotes are parabolas (displaced with each other along the horizontal axis) and that

$$H''_\eta(\pm\infty) \;=\; \frac{1}{2(2\alpha+\beta+\gamma)}\,,$$

in agreement with Eq. (44) since $\partial_\nu^2 E_{\nu,0}$ tends to zero at large $|\nu|$. Recall that we have calculated the central-limit inverse variance $H''_\eta(\lambda)$ before, see Eq. (45), so that around the minimum,

$$H_\eta(\eta/t) \;=\; \frac{1}{4}\,\frac{(\frac{\eta}{t}-\lambda)^2}{2\alpha+\beta+\gamma\;-\;(\alpha+\gamma)\partial_\nu^2 E_{\nu,0}(k^2)|_{\nu=0}} \;+\; \mathcal{O}(|\tfrac{\eta}{t}-\lambda|^3)$$

with $\partial_\nu^2 E_{\nu,0}$ given by Eq. (46). The difference between $H''_\eta(\pm\infty)$ and $H''_\eta(\lambda)$ attests to the non-Gaussian character of the large deviations of η. Fig. 5 presents the behavior of $\frac{1}{\tau}H''_\eta(\pm\infty)$ and of $\frac{1}{\tau}H''_\eta(\lambda)$ for $\wp = 0,\ 0.7$ and 1 as functions of the anisotropy degree $\kappa = \alpha\tau$.

The first quantity diminishes with growing κ from the isotropic value $\frac{1}{2}$ for $\kappa = 0$ to the extremely anisotropic one equal to $\frac{1}{4}$ for $\kappa = 1$ whereas the second one increases starting from the same initial value. It is plausible that for $\wp = 1$, $\frac{1}{\tau}H''_\eta(\lambda)$ diverges in the limit $\kappa \to 1$. We infer that the undimensionalized central-limit covariance $\frac{1}{\tau}H''_\eta(\lambda)$ increases with anisotropy and that the probability of large values of η/t decreases slower than if the rate function H_η were quadratic with the H''_η equal to its central-limit value. Fig. 6 represents the graph H_η

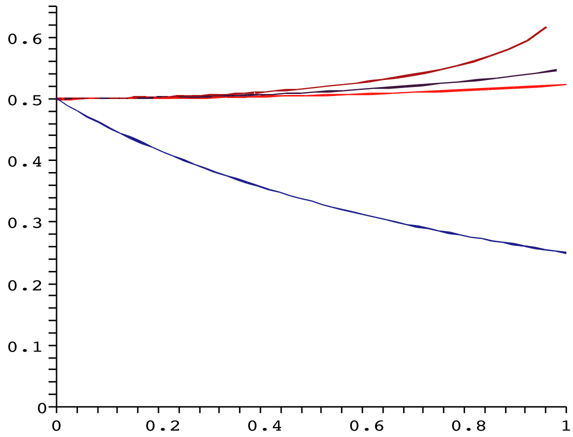


Fig. 5. (Color online) From bottom to top: $\frac{1}{\tau} H''_\eta(\pm\infty)$ (the lower curve) and $\frac{1}{\tau} H''_\eta(\lambda)$ for $\wp = 0,\ 0.7$ and 1 as functions of the anisotropy degree κ.

together with the large value asymptotes and the quadratic approximation near the minimum for $\kappa = 0.8$ and $\wp = 0.9$.

4.8. Time Scales of the Large Deviations Regime

It is interesting to look at the time scales at which the large deviation regime for the difference ρ or η of the stretching exponents sets in. There were three

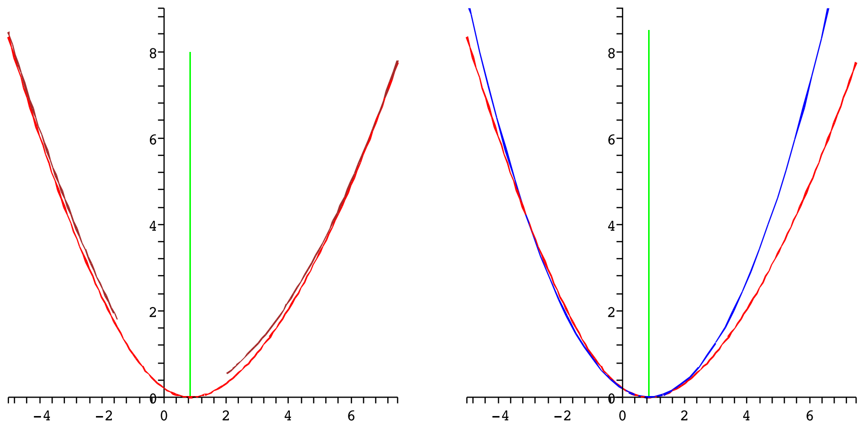


Fig. 6. (Color online) $\tau H_\eta(x/\tau)$ as a function of x with the large η asymptotes on the left figure and the quadratic approximation around the minimum (the curve tighter for large values) for $\kappa = 0.8$ and $\wp = 0.9$. The vertical lines correspond to $\eta/t = \lambda$.

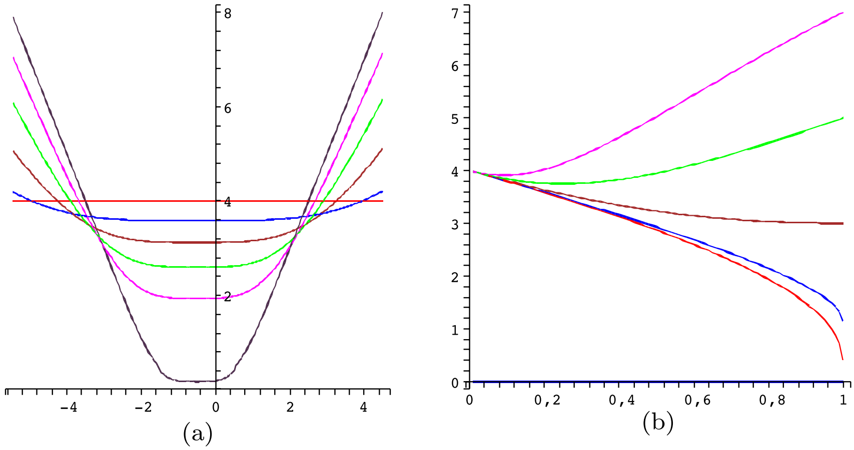


Fig. 7. (Color online) The gap in the spectrum of $\mathcal{H}_\nu$: (a) as a function of ν for, from middle top to bottom at $\nu = 0$, $k^2 = 0,\ 0.2,\ 0.4,\ 0.6,\ 0.8$ and 1, (b) as a function of $k^2 = \frac{\alpha}{\alpha+\gamma}$ for, from top to bottom, $\nu = 4,\ 3,\ 2,\ 2$ and 0.

approximations involved in reducing the exact PDFs to their large deviation form. Let us analyze them one by one.

The first approximation consisted in replacing the PDF $P_t(\phi, \rho)$ by $P_t^{as}(\phi, \rho)$ involving the asymptotic form $\mathcal{L}_{\phi\rho}^{as}$ of the generator $\mathcal{L}_{\phi\rho}$. The asymptotic form should set in exponentially fast in time with the rate given by the difference λ of the Lyapunov exponents. This approximation becomes exact when one analyzes the difference η of the stretching exponents along the unstable flags.

The second approximation consisted of considering only the contributions of the ground state of $\mathcal{H}_\nu$ to the kernel of the operator $\mathrm{e}^{-2(\alpha+\gamma)t\mathcal{H}_\nu}$ in Eq. (39) with the contour integral along $\mathcal{C} = \{Re\,\nu = \nu_{\max}\}$. The contributions of the excited states to that kernel decouple exponentially fast with a rate given by the spectral gap of $2(\alpha+\gamma)\mathcal{H}_{\nu_{\max}}$ for $\nu_{\max}$ equal to the value of ν that maximizes the right hand side of Eq. (42). We obtain this way a continuum of time scales that depend on ρ/t or on η/t. The plot of the gap $E_{\nu,1} - E_{\nu,0}$ of $\mathcal{H}_\nu$ as a function of ν for $k^2 = 0,\ 0.2,\ 0.4,\ 0.6,\ 0.8$ and 1 is given in Fig. 7a. For $k^2 = 1$ the gap is expected to vanish for $-1 \le \nu \le 0$ and its small but positive value on the graph is due to the slowdown in the numerical algorithm. Indeed, as noticed before, the gap of $\mathcal{H}_\nu$ for $\nu = 0$ or $\nu = -1$ is equal to $\pi^2 \boldsymbol{K}(k)^{-2}$ and tends to zero when k^2 tends to 1 (i.e. when γ approaches 0). For $\nu < -1$ or $\nu > 0$, the limiting value of the gap when k^2 tends to 1 should, however, be positive and increasing with $|\nu + 1/2|$. Fig. 7b presents the gap as a function of k^2 for $\nu = 0,\ 1,\ 2,\ 3$ and 4. Since the interval $|\nu_{\max} + 1/2| \le 1/2$ corresponds to $\rho/t \le \lambda$ or to $|\eta/t| \le \lambda$, we infer that the decoupling of the contributions of the excited states of $\mathcal{H}_\nu$ to the PDFs of ρ or η on those intervals takes more and more time when $\gamma \to 0$.

The third approximation in extracting the large deviation form of $P_t(\boldsymbol{\rho})$ or $Q_t(\boldsymbol{\eta})$ consisted in replacing the integral in (41) along the contour $\mathcal{C} = \{Re\,\nu = \nu_{\max}\}$ by its saddle point value. This induces the correction to $H_\rho(\rho/t)$ whose leading term comes from the one-loop contribution

$$\frac{1}{2t}\,\ln\Big(\frac{2\alpha+\beta+\gamma\,-\,(\alpha+\gamma)\,\partial_\nu^2 E_{\nu,0}|_{\nu=\nu_{\max}}}{2\alpha+\beta+\gamma\,-\,(\alpha+\gamma)\,\partial_\nu^2 E_{\nu,0}|_{\nu=0}}\Big)\;=\;\frac{1}{2t}\,\ln\big(H''(\lambda)/H''(\rho/t)\big)$$

and similarly for $H_\eta(\eta/t)$. As we noticed at the end of the previous subsection, it is plausible that for $\wp = 1$, $H_\rho''(\lambda) = H_\eta''(\lambda)$ diverges in the limit $\kappa \to 1$ causing the divergence of the last correction.

4.9. The Case with Equal Lyapunov Exponents

At the end, let us consider the special case of the potential flow with maximal anisotropy when $\wp = 1$ and $\kappa = 1$, i.e. when $\beta = \gamma = 0$. In this case the strain matrix $(S_t^\omega)^i{}_k$ is diagonal with $(S_t^\omega)^1{}_2 = (S_t^\omega)^2{}_1 = 0$ and

$$\big\langle\big(S_t^\omega\big)^1{}_1\,\big(S_{t'}^\omega\big)^1{}_1\big\rangle\;=\;2\alpha\,\delta(t-t') = \big\langle\big(S_t^\omega\big)^2{}_2\,\big(S_{t'}^\omega\big)^2{}_2\big\rangle,\qquad \big\langle\big(S_t^\omega\big)^1{}_1\,\big(S_{t'}^\omega\big)^2{}_2\big\rangle\;=\;0\,.$$

The solution of the multiplicative Itô stochastic equation with initial condition $W_0 = Id$ takes the form

$$W_t\;=\;\mathrm{diag}\Big[\mathrm{e}^{\int_0^t (S_s^\omega)^1{}_1\,ds\,-\,\alpha t},\;\mathrm{e}^{\int_0^t (S_s^\omega)^2{}_2\,ds\,-\,\alpha t}\Big]$$

and the stretching exponents $\boldsymbol{\rho}_t$ are given by the formula

$$\rho_{1t} = \max\,\big(\mathrm{e}^{\int_0^t S_\omega(s)^1{}_1\,ds\,-\,\alpha t},\;\mathrm{e}^{\int_0^t S_\omega(s)^2{}_2\,ds\,-\,\alpha t}\big)$$

$$\rho_{2t} = \min\,\big(\mathrm{e}^{\int_0^t S_\omega(s)^1{}_1\,ds\,-\,\alpha t},\;\mathrm{e}^{\int_0^t S_\omega(s)^2{}_2\,ds\,-\,\alpha t}\big)\,.$$

This results in the joint time t PDF

$$P_t(\rho_1,\rho_2)\;=\;\frac{1}{2\pi\alpha t}\,\mathrm{e}^{-\frac{t}{4\alpha}\left(\frac{\rho_1}{t}+\alpha\right)^2}\,\mathrm{e}^{-\frac{t}{4\alpha}\left(\frac{\rho_2}{t}+\alpha\right)^2}\,\theta(\rho_1-\rho_2)$$

or, in terms of $r \equiv \frac{1}{2}(\rho_1+\rho_2)$ and $\rho \equiv \rho_1 - \rho_2$,

$$P_t(\rho_1,\rho_2)\;=\;\frac{1}{2\pi\alpha t}\,\mathrm{e}^{-\frac{t}{2\alpha}\left(\frac{r}{t}+\alpha\right)^2}\,\mathrm{e}^{-\frac{t}{8\alpha}\left(\frac{\rho}{t}\right)^2}\,\theta(\rho).$$

Similarly the stretching exponents $\boldsymbol{\eta}_t$ along the flags $F = OF^0$, are given by the Iwasawa decomposition (48) of the matrices $\tilde{W} = W_t O$ with the flags $F = OF^0$ distributed with respect to the measure $\chi(dF)$ on the flag variety Fl

that is invariant under the process W_t, see Eq. (16). Any such measure has to be concentrated on the two flags given by the coordinate axis. On obtains then the formula

$$\eta_{1t} \;=\; \mathrm{e}^{\int_0^t S_\omega(s)^1{}_1\, ds \,-\, \alpha t}\,, \qquad \eta_{2t} \;=\; \mathrm{e}^{\int_0^t S_\omega(s)^2{}_2\, ds \,-\, \alpha t}$$

if OF^0 is given by the first coordinate axis or the one with interchanged η_{i_t} if OF^0 is given by the second coordinate axis. In any case, the joint t PDF of the stretching exponents $\boldsymbol{\eta}$ takes the form

$$P_t(\eta_1, \eta_2) \;=\; \frac{1}{4\pi\alpha t}\; \mathrm{e}^{-\frac{t}{4\alpha}\left(\frac{\eta_1}{t}+\alpha\right)^2}\; \mathrm{e}^{-\frac{t}{4\alpha}\left(\frac{\eta_2}{t}+\alpha\right)^2}$$

or, in terms of $r \equiv \frac{1}{2}(\eta_1 + \eta_2)$ and $\eta \equiv \eta_1 - \eta_2$,

$$P_t(\eta_1, \eta_2) \;=\; \frac{1}{4\pi\alpha t}\; \mathrm{e}^{-\frac{t}{2\alpha}\left(\frac{r}{t}+\alpha\right)^2}\; \mathrm{e}^{-\frac{t}{8\alpha}\left(\frac{\eta}{t}\right)^2}\,.$$

The PDF of r_t agrees with that of Eq. (27), i.e. r_t/t is the normal variable with mean $-\alpha$ equal to the half of the sum of the Lyapunov exponents and with variance α/t. As for ρ_t/t, it is distributed as an absolute value of the centered normal variable with variance $4\alpha/t$. In particular, the difference of the Lyapunov exponents vanishes and the large deviation rate function for ρ_t is quadratic:

$$H_\rho(\rho/t) \;=\; \frac{1}{8\alpha}\left(\frac{\rho}{t}\right)^2. \tag{53}$$

Similarly, η/t is a normal variable with mean zero and variance $4\alpha/t$ and

$$H_\eta(\eta/t) \;=\; \frac{1}{8\alpha}\left(\frac{\eta}{t}\right)^2. \tag{54}$$

The values of the Lyapunov exponents agree with those given by the limiting values of Eqs. (28) and (36). Recall however, that the non-Gaussianity of the large deviations, as measured by the difference $\frac{1}{\tau}[H_\eta''(\lambda) - H_\eta''(\pm\infty)]$, was increasing with the growth of the anisotropy degree κ, see Fig. 5. For $\wp = 1$, in particular, $\frac{1}{\tau}H_\eta''(\lambda)$ was growing with κ whereas $\frac{1}{\tau}H_\eta''(\pm\infty) = \frac{1}{2(\kappa+1)}$ decreased to the value $\frac{1}{4}$ for $\kappa = 1$. This seems in contradiction with the results (53) and (54) with the quadratic large deviations rate functions for $\wp = 1$ and $\kappa = 1$ with $H_\rho''(\rho/t) = H_\eta''(\eta/t) = \frac{\tau}{4}$ everywhere and not only at infinity. The solution of the puzzle lies in the non-uniformity of the large deviation regime when $\beta, \gamma \to 0$ and the two Lyapunov exponents tend to each other. As we have noticed in the previous subsection, the time scales at which the large deviation regime sets in diverge when $\gamma \to 0$ (and, consequently, $\beta \to 0$ and $\lambda \to 0$). That could explain why the limit of $H_\rho''(\lambda) = H_\eta''(\lambda)$ when $\gamma \to 0$ is not equal to the value of $H_\rho''(0) = H_\eta''(0)$ for $\gamma = 0$.

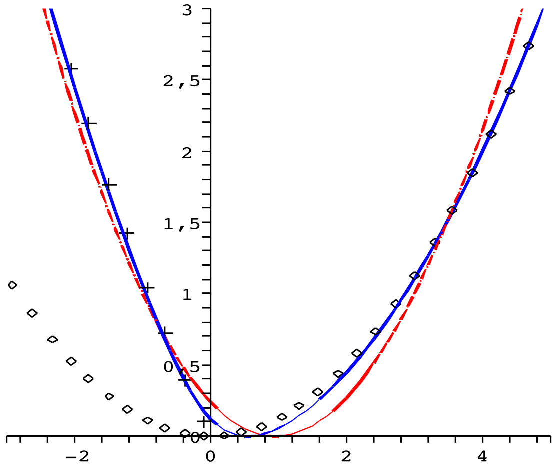


Fig. 8. (Color online) Illustration of conjectured point-wise convergence of $\tau H_\eta(x/\tau)$ to $x^2/8$ (diamonds) for $x > 0$ and to $x^2/8 - x$ (crosses) for $x < 0$. The dotted line corresponds to $\beta = 0$ and $\gamma = \alpha$, the solid one to $\beta = 0$ and $\gamma = 0.01\alpha$.

Numerical calculations, see Fig. 8, seem to indicate, however, that H_ρ still converges point-wise to its form for $\gamma = 0$ when $\gamma \to 0$. Such point-wise convergence when $\gamma \to 0$ cannot take place for the large deviations rate function H_η. Indeed, recall that that for $\gamma > 0$ it is $H_\eta(\eta/t) + \eta/(2t)$ that is an even function of η/t whereas for $\gamma = 0$, the rate function $H_\eta(\eta/t)$ is even itself. The point-wise convergence of H_η that is a function on the whole real line cannot then hold. Instead, for negative η/t, the rate function $H_\eta(\eta/t)$ should converge when $\gamma \to 0$ to $H_\eta(\eta/t) - \eta/t$. Let us note that the evenness of $H_\eta(\eta/t) + \eta/(2t)$ for $\gamma > 0$ is a consequence of the relations

$$\mathcal{P}_t(\phi, \eta) \equiv \mathrm{e}^{t\,\mathcal{L}_{\phi\eta}}(0, 0; \eta, \phi) \;=\; \mathrm{e}^{\eta}\,\mathrm{e}^{t\,\mathcal{L}_{\phi\eta}}(\phi, \eta; 0, 0) \;=\; \mathrm{e}^{\eta}\,\mathrm{e}^{t\,\mathcal{L}_{\phi\eta}}(\phi, 0; 0, -\eta)\,. \tag{55}$$

The first one follows from the self-adjointness of the operator $\mathcal{L}_{\phi\eta}$ with respect to the measure $\mathrm{e}^{\eta}\,d\phi d\eta$ on the product of the circle by the real line and the second one from the commutation of $\mathcal{L}_{\phi\eta}$ with the translations of η. Recall that the PDF of η is given by the integral

$$\int_0^{\pi} \mathrm{e}^{t\,\mathcal{L}_{\phi\eta}}(\phi, \eta)\, d\phi$$

into which the initial and the final angles do not enter in a symmetric way so that the equalities (55) *do not* imply that

$$\int_0^\pi \mathrm{e}^{t\,\mathcal{L}_{\phi\rho}}(0,0;\phi,\rho)\,d\phi \;=\; \mathrm{e}^{\eta}\int_0^\pi \mathrm{e}^{t\,\mathcal{L}_{\phi\eta}}(0,0;\phi,-\eta)\,d\phi\,.$$

Nevertheless, for $\gamma > 0$ the last equality holds if the full PDF of η are replaced by its large deviation approximation, the angular asymmetry showing up only in the prefactors related to the ground state eigenfunctions of $\mathcal{H}_\nu$. On the other hand, for $\gamma = 0$ the angular asymmetry does not decouple from the large deviation form of the PDF of η and conspires to render the latter even. The lack of point-wise convergence of H_η to its form for $\gamma = 0$ is a reflection of the singular behavior of the eigenfunctions of $\mathcal{H}_\nu$ when $\gamma \to 0$.

To summarize, although the distribution of the stretching exponents $\boldsymbol{\eta}$ still exhibits large deviation regime when $\gamma = 0$, the corresponding rate function is not equal to the limit of the rate functions for $\gamma > 0$ signaling that when two Lyapunov exponents coincide, the occurrence of the multiplicative large deviation regime becomes problematic.

5. CONCLUSIONS

We have examined in detail the tangent process W_t^ω describing the evolution of infinitesimal separations between Lagrangian trajectories in the two-dimensional Kraichnan flow in a periodic square. The process W_t^ω is driven by the time decorrelated strain whose distribution is, in general, anisotropic, possessing only the symmetry with respect of the 90^o rotations and axes reflections. Our interest was concentrated on the large deviation regime of the stretching exponents ρ_i or η_i that appear in the matrix decompositions $W = O'\,\mathrm{diag}[\mathrm{e}^{\rho_1}, \mathrm{e}^{\rho_2}]\,O$ or $W = O'\,\mathrm{diag}[\mathrm{e}^{\eta_1}, \mathrm{e}^{\eta_2}]\,N$ with orthogonal matrices O, O' and upper-triangular N. The anisotropy couples the dynamics of the stretching exponents to the evolution of the matrices O', in contrast to the situation in the isotropic case where the stochastic dynamics of the stretching exponents is decoupled from that of the matrices O' and O or O' and N. The stochastic evolution of the matrices O' becomes, however, independent, at least at long times, from that of the stretching exponents, attaining exponentially fast a stationary state. The latter feeds to the evolution of the stretching exponents in a steady fashion permitting them still to attain the large deviation regime. The large deviation rate function for the stretching exponents may be expressed in terms of the ground state energy of an operator on the group of orthogonal matrices parametrized by the variables conjugate to the stretching exponents. The contribution of the excited states to the PDF of the stretching exponents decouples exponentially fast with the rate equal to the gap of the angular operator. This scenario for the multiplicative large deviations seems quite general whenever the Lyapunov exponents are all different, at least in the homogeneous Kraichnan model. When some of the Lyapunov exponents become close, some of the time scales for the appearance of the large deviation regime as well as the prefactors multiplying the exponential large deviation PDF may diverge.

What is special about the flow on the periodic square is that the operator on the orthogonal group in question takes form of the integrable periodic Schrödinger operator of the Lamé type facilitating the calculations. This simplification due to the hidden integrable structure allowed to obtain closed formulae for the Lyapunov exponents in terms of elliptic integrals and to analyze the large deviation rate function with precision. The results of the analysis show that the anisotropy effects lower the top Lyapunov exponent and increase the lower one (relative to an overall inverse time scale), with the sum of the two fixed by the compressibility degree of the flow. The sum of the two stretching exponents is normally distributed with the covariance again fixed by the compressibility degree. The difference of the stretching exponents, however, exhibits in the presence of anisotropy non-Gaussian large deviations. Its central limit covariance grows with anisotropy but the quadratic large-value asymptotes of the rate function have the top coefficients that decrease with increasing anisotropy. This non-Gaussian scenario for the multiplicative large deviations applies, however, only when the Lyapunov exponents are different. At the extreme anisotropy, when the strain matrix is diagonal with independent equally distributed entries, the two Lyapunov exponents coincide and the large deviations for the stretching exponents are Gaussian. We have analyzed in more detail this discontinuous restoration of the Gaussianity of large deviations.

For the Kraichnan flow in a periodic rectangle, the multiplicative large deviations may be analyzed similarly. The results will be published elsewhere. It remains an open question whether the Kraichnan flow in a three-dimensional periodic box possesses a hidden integrable structure that would permit to extend the analysis of the present paper to that case.

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